WEAKLY CHAINABLE CIRCLE-LIKE CONTINUA

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Gary A. Feuerbacher

August, 1974

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ABSTRACT

This paper investigates the problem of ascertaining which circle-like continua are continuous images of chainable continua. In the second chapter, the notion of the "revolving number" of a map from S^1 onto S^1 is introduced and used to classify the planar, non-chainable, circle-like continua by structure: decomposable; "self-entwined" (a notion introduced in chapter 2); indecomposable, non-self-entwined. The main theorem in chapter 3 is a characterization of weakly chainable circle-like continua; the classification scheme of chapter 2 is used to prove this result.

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CHAPTER I

INTRODUCTION

Suppose that for each positive integer i, X_i is a compact metric space and f_i^{i+1} is a map (continuous function) from X_{i+1} onto X_i. Let M be the subset of the Cartesian product space $\prod_{i=1}^{\infty} X_i$ consisting of the set of all sequences p such that for each i, p_i is in X_i and $f_i^{i+1}(p_{i+1}) = p_i$. Then M, with the relative topology from $\prod_{i=1}^{\infty} X_i$, is called the inverse limit of the inverse system (X_i, f_i^{i+1}), and denoted $\varprojlim (X_i, f_i^{i+1})$. If $m > n$, f_n^m will denote the composition of the maps f_n^{n+1}, f_{n+1}^{n+2} , ... , f_{m-1}^m ; f_m^m will denote the identity function on X_m. For each positive integer i, PR_i will denote the natural projection of M onto X_i; i.e., $PR_i(a_1, a_2, a_3, \ldots) = a_i$. The theorems in this paper are concerned with inverse limits in which each factor space X_i is a circle (i.e., circle-like continua), and in which each factor space X_i is an arc (i.e., arc-like, or chainable, continua).

The following theorems will be used frequently:

A. If $(n_1, n_2, n_3, \ldots)$ is an increasing sequence of positive integers, then M is homeomorphic to the inverse limit of the inverse system $(X_{n_i}, f_{n_i}^{n_{i+1}})$.

(In Theorems B and C, assume that $K = \varprojlim (Y_i, g_i^{i+1})$.)

B. Suppose h is a sequence of maps such that (1) for each positive integer i, h_i is a map from X_i onto Y_i, and (2) for each i, $h_i \circ f_i^{i+1} = g_i^{i+1} \circ h_{i+1}$. Then the function G from M into $\prod_{i=1}^{\infty} Y_i$ defined by $G(a_1, a_2, a_3, \ldots) = (h_1(a_1), h_2(a_2), \ldots)$ is a map from M onto K.

Definition. (see [1]) Suppose each of A and B is a metric space and each of u and v is a map from A into B. Suppose $c > 0$. The statement that $u \underset{c}{=} v$ means that for each point x in A, $\mathrm{dist}_B(u(x),v(x)) < c$.

C. (Theorem 3 of [1]) Suppose e is a decreasing sequence of positive numbers with sequential limit 0. Suppose h is a sequence of maps such that (1) for each positive integer i, h_{2i} is a map from Y_{2i} onto X_{2i} and h_{2i-1} is a map from X_{2i-1} onto Y_{2i-1} ; (2) for each triple (i,j,k) of positive integers with $i < j$ and $k \leq 2i-1$,

$$g_k^{2i-1} \circ h_{2i-1} \circ f_{2i-1}^{2j-1} \underset{e_{2i-1}}{=} g_k^{2j-1} \circ h_{2j-1}$$

and $$g_k^{2i-1} \circ h_{2i-1} \circ f_{2i-1}^{2j-2} \circ h_{2j-2} \underset{e_{2i-1}}{=} g_k^{2j-2} \;;$$

(3) for each triple (i,j,k) of positive integers with $i < j$ and $k \leq 2i$,

$$f_k^{2i} \circ h_{2i} \circ g_{2i}^{2j} \underset{e_{2i}}{=} f_k^{2j} \circ h_{2j}$$

and $$f_k^{2i} \circ h_{2i} \circ g_{2i}^{2j-1} \circ h_{2j-1} \underset{e_{2i}}{=} f_k^{2j-1} .$$

Then M is homeomorphic to K. In case $X_i = Y_i$ and h_i is the identity map for each i, it suffices that for each ordered triple (i,j,k) of positive integers with $k \leq i < j$,

$$g_k^i \circ f_i^j \underset{e_i}{=} g_k^j$$

and $$f_k^i \circ g_i^j \underset{e_i}{=} f_k^j$$

for M to be homeomorphic to K.

CHAPTER II

STRUCTURE OF CIRCLE-LIKE CONTINUA

In [2], Bing characterized the class of non-planar circle-like continua, and in [3], Ingram characterized the chainable circle-like continua. In this chapter, the class of non-chainable, planar, circle-like continua is subdivided into three subclasses: the decomposable; the self-entwined (a concept to be introduced in this chapter); the indecomposable, non-self-entwined. This classification scheme is used to prove the main result of chapter III.

The "circle", S^1, is the unit circle on the complex plane. If P and Q are two non-antipodal points of the circle, and L the length (in the usual metric) of the minor arc between them, then the distance from P to Q, denoted $|P-Q|$, is defined as $\frac{L}{2\pi}$. The distance between antipodal points is ½. The "wrapping function", denoted ϕ, is the map from the real line onto S^1 which sends the number x to $e^{2\pi ix}$. Let S^1 be oriented so that ϕ is order-preserving. If A and B are points of S^1, then the arc $[A,B]$ of S^1 is the ϕ-image of an interval $[a,b]$, $b-a < 1$, with $\phi(a) = A$ and $\phi(b) = B$. If C is a point of S^1, then we write $A < C < B$ in case there is a number c, $a < c < b$, with $\phi(c) = C$. Notice that if $b-a \le \frac{1}{2}$, then $|A-B| = b-a$.

Definition. If f is a map from S^1 into S^1, then the degree of f, denoted deg f, is that integer n such that f is homotopic to the n-th power of the complex identity function restricted to S^1.

The next two definitions are modifications of concepts developed by J.T. Rogers in [4], approached here from a homotopy-theoretic rather than combinatorial point of view.

Suppose f is a map from S^1 onto S^1, and deg f ≥ 0.

Definition. Suppose T is an arc in S^1. Let u be a lift of $f|T$, i.e., u is a map from T into the real line, and $f|T = \phi \circ u$. Then deg(T,f) is defined as diam u(T); this number is independent of which lift map is taken.

In case deg(T,f) is an integer, deg(T,f) is the number of times the arc T is "wrapped around" the circle by f.

Lemma 1. Suppose D is the number set to which a number r belongs if and only if there is an arc Q in S^1 such that r = deg(Q,f). Then D is bounded above.

Proof. Since f is uniformly continuous, let $d > 0$ be such that any d-ball in S^1 is mapped into a semi-circle in S^1. Let m be an integer greater than $\frac{1}{2d}$. If A is an arc in S^1, then A may be covered by a linear chain of d-balls with no more than m links, implying that deg(A,f) $\leq \frac{m}{2}$.

Definition. Suppose D is as in the hypothesis of Lemma 1. The revolving number of f, denoted R(f), is sup D.

Lemma 2. Suppose P and Q are points of S^1. Let T be a point sequence with each value in the interior of the arc [Q,P], and T converges to P. Let u be a sequence of maps such that for each positive integer i, u_i is a lift of $f|[P,T_i]$, and $u_i(P) = u_1(P) = Z$. Then

$\lim\limits_{i \to \infty} u_i(T_i) = Z + \deg f$.

Proof. Suppose deg f = n. The quotient map $\frac{f}{I^n}$ is inessential. Let v be a lift of $\frac{f}{I^n}$. Then $\frac{f}{I^n} = \phi \circ v$, and $f = I^n \cdot (\phi \circ v)$. Let $e > 0$. Since T converges to P, and v is continuous, let N be a positive integer such that if $m \geq N$, then $|T_m - P| < \frac{e}{2n+1}$ and $|v(T_m) - v(P)| < \frac{e}{2}$.

Suppose $m \geq N$. Let [a,b] be an interval, $b-a < 1$, such that $\phi(a) = P$

and $\phi(b) = T_m$. Let h be the inverse map of $\phi|[a,b]$. If x is in $[P,T_m]$, then $x^n = [\phi(h(x))]^n = \phi(nh(x))$. Hence $\phi \circ u_m = f|[P,T_m]$ $= (\phi \circ (nh)) \cdot (\phi \circ v) = \phi \circ (nh + v)$. There is an integer J such that $u_m + J = nh + v$. Thus $u_m(T_m) - u_m(P) = nh(T_m) - nh(P) + v(T_m) - v(P)$ $= n(b-a) + v(T_m) - v(P)$. Since $|T_m - P| < \frac{e}{2n+1}$, $1 - \frac{e}{2n+1} < b-a < 1$, and $n - \frac{e}{2} < n - \frac{ne}{2n+1} \leq n(b-a) \leq n$. Also $-\frac{e}{2} < v(T_m) - v(P) < \frac{e}{2}$, hence

$$n - e < n(b-a) + v(T_m) - v(P) < n + \frac{e}{2}$$

$$n - e < \quad u_m(T_m) - u_m(P) \quad < n + \frac{e}{2}$$

$$-e < \quad u_m(T_m) - (Z + n) \quad < \frac{e}{2}.$$

This completes the proof.

Lemma 2 yields immediately $R(f) \geq \deg f$.

Using the results of Ingram in [3] and of McCord (page 29 of [5]), we have

Theorem D. If C is a circle-like continuum, then C is planar and non-chainable if and only if C is homeomorphic to $\underleftarrow{\text{Lim}}\,(X_i, f_i^{i+1})$, in which each X_i is S^1, and $\deg f_i^{i+1} = 1$ for each i.

Notation. "p.n.c.c.l." will mean "planar, non-chainable, circle-like".

We are ready to prove the main result of this chapter.

Definition. Suppose M is a p.n.c.c.l. continuum as in Theorem D. Then M is said to be in class 1 if, for each positive integer i, there exists a number $Z_i, 1 \leq Z_i < 2$, such that for each positive integer j, $R(f_i^{i+j}) \leq Z_i$. We say that M is in class 2 if for each i, and each number y, $1 \leq y < 2$, there is j such that $R(f_i^{i+j}) > y$. Similarly, M is in class A if, for each i, there exists Z_i, $1 \leq Z_i < 3$, such that for each positive integer j, $R(f_i^{i+j}) \leq Z_i$; also, M is in class B if for each i, and each y, $1 \leq y < 3$, there is j such that $R(f_i^{i+j}) > y$.

Theorem 1. Suppose M is a p.n.c.c.l. continuum. Then either M is homeomorphic to a member of class 1 or M is homeomorphic to a member of class 2. Furthermore, either M is homeomorphic to a member of class A or M is homeomorphic to a member of class B.

Proof. Let $M = \varprojlim (X_i, f_i^{i+1})$ as in Theorem D. Suppose M is not in class 2. Then there is a number Z, $1 \leq Z < 2$, and there is a positive integer i such that for each j, $R(f_i^{i+j}) \leq Z$. Let D be the set of all ordered pairs (p,y) such that p is a positive integer, y is a number, $1 \leq y < 2$, and for each positive integer j, $R(f_p^{p+j}) \leq y$.

Case (1). The domain of the relation D is bounded. Let K be an integer greater than every element in the domain of D. Let Z be a number, $1 \leq Z < 2$, and i be a positive integer. Then $(K+i,Z)$ is not in D. Thus there is j such that $R(f_{K+i}^{K+i+j}) > Z$. Let $C = \varprojlim (X_{K+i}, f_{K+i}^{K+i+1})$. Then C is in class 2, and M is homeomorphic to C by Theorem A.

Case (2). The domain of D is not bounded. Let $(n_1, n_2, n_3, \ldots)$ be an increasing sequence of positive integers whose range is the domain of D. Let h be a function whose domain is the domain of D, and h is a subset of D. Let $C = \varprojlim (X_{n_i}, f_{n_i}^{n_{i+1}})$. Then C is in class 1. For: if i is a positive integer, then $h(n_i)$ is a number, $1 \leq h(n_i) < 2$, such that for each j, $R(f_{n_i}^{n_{i+j}}) \leq h(n_i)$. By Theorem A, M is homeomorphic to C. The second assertion of Theorem 1 is proved similarly.

Trivially, class B is a subset of class 2. The collection of all p.n.c.c.l. continua is class 1 $\cup$ class B $\cup$ (class 2 $\setminus$ class B). We will see that if M is a p.n.c.c.l. continuum, then M is indecomposable if and only if M is homeomorphic to a member of class 2.

Definition. The continua which are homeomorphic to members of class B will be called self-entwined(this notion is also a modified version of an idea in [4]).

We will see that the self-entwined continua have some of the properties of non-planar circle-like continua (e.g., Corollary to Lemma 8; Theorem 5).

Assume, as before, that f is a map from S^1 onto S^1, and deg $f \geq 0$.

Definition. If A is an arc in S^1, and t is a lift of $f|A$, then there is a subarc B of A such that the map t sends the endpoints of B to the endpoints of the interval t(A). An arc with this property of B will be called "type 1".

Lemma 3. If $R(f) >$ deg f, then there is an arc D in S^1 such that $\deg(D,f) = R(f)$.

Proof. Let A be a sequence of arcs in S^1 such that (1) each value of A is of type 1; (2) for each i, $\deg(A_i,f) \leq \deg(A_{i+1},f)$; (3) $\deg(A,f)$ converges to $R(f)$; (4) letting $A_i = [P_i,Q_i]$, the point sequence P converges to a point c, and Q converges to a point d. Let L be the limiting set of A. Suppose c = d. Then $L = S^1$; otherwise, $\lim_{i \to \infty} \operatorname{diam} A_i = 0$, and $\lim_{i \to \infty} \deg(A_i,f) = 0 \neq R(f)$. Let y and z be points of S^1, with $y < c < z$. There is a positive integer m such that, for $j \geq m$, $y < P_j < z$, and $P_j < z < y < Q_j$. Let v be a lift of $f|[y,z]$. Let u be a sequence of maps such that for each i, u_i is a lift of $f|A_i$, and for $j \geq m$, $u_j(z) = v(z)$. If $j \geq m$, then $u_j|[P_j,z] = v|[P_j,z]$. Thus $\lim_{i \to \infty} u_i(P_i) = \lim_{i \to \infty} v(P_i) = v(c)$. By an argument similar to that of Lemma 2, $\lim_{i \to \infty} u_i(Q_i) = v(c) + \deg f$. Hence $\lim_{i \to \infty} (u_i(Q_i)-u_i(P_i)) = \deg f$.

But each A_i is of type 1, and $\deg(A_i,f) = |u_i(Q_i) - u_i(P_i)|$. Therefore $\lim_{i\to\infty} \deg(A_i,f) = \deg f < R(f)$, a contradiction.

We have $c \neq d$, and $L = [c,d]$. Let w be a lift of $f|[c,d]$. Let x be a point in the interior of $[c,d]$. There is a positive integer m such that for $j \geq m$, $P_j < x < Q_j$. As in the preceding paragraph, let, for each i, u_i be a lift of $f|A_i$, and for $j \geq m$, $u_j(x) = w(x)$. We have (similar to the previous paragraph) $\lim_{i\to\infty} u_i(P_i) = w(c)$, and $\lim_{i\to\infty} u_i(Q_i) = w(d)$. Thus $\lim_{i\to\infty} |u_i(Q_i) - u_i(P_i)| = |w(d) - w(c)|$, and $R(f) = \deg([c,d],f)$.

Definition. If P is an arc in S^1, P is of type 1, and $\deg(P,f) = R(f)$, then P is called a defining arc for R(f).

Lemma 4. Suppose $\deg f \geq 1$ and $[a,b]$ is a defining arc for R(f). Let t be a lift of $f|[a,b]$. Then $t(a) < t(b)$, and $\deg([b,a],f) = R(f) - \deg f$.

Proof. Suppose $t(b) < t(a)$. By Lemma 2, let c be a point, $b < c < a$, such that if u is a lift of $f|[a,c]$, and $u(a) = t(a)$, then $-\frac{1}{2} < t(a) + \deg f - u(c) < \frac{1}{2}$. If u is such a map, then

$$t(a) - t(b) + \deg f - \tfrac{1}{2} < u(c) - t(b), \text{ and}$$

$R(f) + \frac{1}{2} \leq R(f) + \deg f - \frac{1}{2} < u(c) - u(b)$. Hence $\deg([b,c],f) > R(f)$, a contradiction.

To prove the second assertion, let v be a lift of $f|[b,a]$, and suppose $v(b) = t(b)$. Lemma 2 yields $v(a) = t(a) + \deg f$, whence $\deg([b,a],f) \geq R(f) - \deg f$. Assume $\deg([b,a],f) > R(f) - \deg f$. No point of $v([b,a])$ is greater than $t(b)$. Suppose c is a point, $b < c < a$, such that $v(c) < t(a) + \deg f$. Let w be a lift of $f|[c,b]$ such that $w(a) = t(a)$. Then $w(c) = v(c) - \deg f < t(a)$. But $w(b) - w(c) = t(b) - w(c) > t(b) - t(a) = R(f)$, a contradiction.

Notice that the arc [b,a] is of type 1: $v([b,a]) = [v(a),v(b)]$.

Theorem 2. If M is a p.n.c.c.1. continuum, then M is indecomposable if and only if M is homeomorphic to a member of class 2.

Proof. Let $M = \underset{\leftarrow}{\mathrm{Lim}}\,(X_i, f_i^{i+1})$ as in Theorem D. Suppose M is in class 2. By a result of Kuykendall ([6, Theorem 2]), M is indecomposable if and only if for each positive integer n, and each number $e > 0$, there are a positive integer j and three points of X_{n+j} such that if K is a subcontinuum of X_{n+j} containing two of them, then $\mathrm{dist}_n(x, f_n^{n+j}(K)) < e$, for each point x in X_n. Suppose n is a positive integer and $\frac{1}{2} > e > 0$. Let j be such that $R(f_n^{n+j}) > 2 - e$. Let [A,B] be a defining arc for $R(f_n^{n+j})$, and t a lift for $f_n^{n+j}|[A,B]$. Then $t([A,B]) = [t(A),t(B)] = [a,b]$, with $b > a + 1$. Let C be a point in [A,B], with $t(C) = a + 1$. Then $[a,a + 1] \subseteq t([A,C])$, and $[a + 1,b] \subseteq t([C,B])$. By Lemma 4, letting v be a lift of $f_n^{n+j}|[B,A]$ such that $v(B) = t(B)$, we have $v([B,A]) = [a+1,b]$. Now, $\phi([a,a + 1]) = S^1$, and $\phi([a + 1,b])$ is either S^1 or an arc of length greater than $1 - e$. Hence $S^1 \setminus \phi([a + 1,b])$ either is not a point set or is an open arc of length less than e. For each point x of S^1, $|x - \phi([a,a + 1])| = 0$, and $|x - \phi([a + 1,b])| < e$. Also, $\phi([a + 1,b]) = \phi(v([B,A])) = f_n^{n+j}([B,A])$; similarly, $\phi([a,a+1]) \subseteq f_n^{n+j}([A,C])$ and $\phi([a + 1,b]) \subseteq f_n^{n+j}([C,B])$. By Kuykendall's theorem, M is indecomposable.

To prove the converse, suppose M is indecomposable. Then for each integer $K \geq 2$, there are K points of M such that M is irreducible between each two of them. A corollary of [6, Theorem 2] is that M being indecomposable implies for each triple (n,p,e), n a positive integer, p an integer, $p \geq 2$, and $e > 0$, there are a positive integer j and p points of X_{n+j}

such that if L is a subcontinuum of X_{n+j} containing two of them, then $\mathrm{dist}_n(x, f_n^{n+j}(L)) < e$, for each point x in X_n. Suppose n is a positive integer and $1 > e > 0$. Let N be an integer such that $N - 2 > \frac{1}{e}$. Let j be a positive integer and W be a set of N points of S^1 such that if A is an arc containing two of them, then $\deg(A, f_n^{n+j}) > 1 - \frac{e}{N-1}$ (similar to the previous paragraph). Let $(p_1, p_2, \ldots, p_N)$ be a reversible sequence of points of S^1, ordered by the orientation of S^1, whose range is the set W. Let v be a lift of $f_n^{n+j}|[p_1, p_N]$. Let, for $1 \le i \le N - 1$, $[a_i, b_i]$ be a subarc of $[p_i, p_{i+1}]$ of type 1. If, for some i, $v([a_i, b_i]) = [v(b_i), v(a_i)]$, then since $v(a_i) - v(b_i) > 1 - \frac{e}{N-1}$, by Lemma 2, $R(f_n^{n+j}) > 2 - \frac{e}{N-1} > 2 - e$. Similarly, if, for some i, $v(b_i) - v(a_{i+1}) > 1 - e$, then $R(f_n^{n+j}) > 2 - e$. Assume that for $1 \le i \le N - 1$, $v(b_i) - v(a_i) > 1 - \frac{e}{N-1}$, and for $1 \le i \le N - 2$, $v(b_i) - v(a_{i+1}) \le 1 - e$; then $v(a_{i+1}) - v(b_i) \ge e - 1$, and $v(a_{i+1}) - v(a_i) > e - \frac{e}{N-1}$. Therefore $v(a_{N-1}) - v(a_1) = \sum_{i=1}^{N-2} (v(a_{i+1}) - v(a_i)) > (N - 2)(e - \frac{e}{N-1})$. But $v(b_{N-1}) - v(a_{N-1}) > 1 - \frac{e}{N-1}$, and $v(b_{N-1}) - v(a_1) > 1 + (N-2)e - e$. Since $(N-2)e > 1$, we have $R(f_n^{n+j}) \ge v(b_{N-1}) - v(a_1) > 2 - e$, whence M is in class 2.

Definition. Suppose g is a map from a continuum X onto a continuum Y. Then g is said to be weakly confluent if, for each subcontinuum K of Y, there is a component C of $g^{-1}(K)$ such that $g(C) = K$.

Lemma 5. If g is a map from a continuum X onto S^1, and g is essential, then g is weakly confluent.

Proof. Suppose g is a map from X onto S^1, and g is not weakly confluent. Then g is inessential. For: Let $[p,q]$ be an arc in S^1 such that no component of $g^{-1}([p,q])$ maps onto $[p,q]$ under g. We may assume that $[p,q]$ is properly contained in a semi-circle; if not, then let h be a homeomorphism from S^1 onto S^1 which sends $[p,q]$ to an arc of length less than $\frac{1}{2}$; if $h \circ g$ is inessential, $h^{-1} \circ h \circ g$ is inessential. Let $W = g^{-1}([p,q])$; Y = the set of all components of W; Y_1 = the set of components of W which contain a point of $g^{-1}(p)$; Y_2 = the set of components of W which contain a point of $g^{-1}(q)$. Now, $Y = Y_1 \cup Y_2$. For: suppose J is an element of $Y \setminus (Y_1 \cup Y_2)$. Then $J \subseteq X \setminus g^{-1}([q,p])$, which is open in X. Let T be the component of $X \setminus g^{-1}([p,q])$ which contains J. Since X is a continuum, $\overline{T}$ contains a point of $g^{-1}([q,p])$. But $\overline{T} \subseteq W$, hence $\overline{T} \subseteq J$, and J contains a point of $g^{-1}([q,p])$, a contradiction.

If K is a sequence of continua lying in W, then K has a subsequence with a sequential limiting set L, and L is also a continuum lying in W. Since $g^{-1}(p)$ and $g^{-1}(q)$ are closed in X, this implies that Y_1^* and Y_2^* are closed sets. Since no component of W contains both a point of $g^{-1}(p)$ and a point of $g^{-1}(q)$, Y_1^* and Y_2^* are mutually exclusive.

Let r be a function from X into S^1 such that if x is in $X \setminus W$, then $r(x) = g(x)$; $r(Y_1^*) = \{p\}$; $r(Y_2^*) = \{q\}$. Since the r^{-1} image of a set closed in S^1 is closed in X, r is continuous. But $r(X) \neq S^1$, thus r is inessential. Since $r \underset{\frac{1}{2}}{=} g$, g is homotopic to r, and g is inessential.

Lemma 6. Suppose each of f and g is a map from S^1 onto S^1, and $\deg g \geq 0$, $\deg f \geq 1$. Then $R(g \circ f) \geq R(g)$.

Proof. In case $R(g) = \deg g$, we have $R(g \circ f) \geq \deg(g \circ f)$ $= (\deg g)(\deg f) \geq \deg g = R(g)$. Suppose $R(g) > \deg g$. Let P be a defining

arc for $R(g)$, and let t be a lift of $g|P$. Since f is essential, thus weakly confluent, let Q be an arc in S^1 such that $f(Q) = P$. Let u be a lift of $(g \circ f)|Q$. Then $(g \circ f)|Q = \phi \circ u$, and $g|P = \phi \circ t$. Hence $g \circ f|Q = \phi \circ t \circ f|Q = \phi \circ u$. Thus
$R(g \circ f) \geq \text{diam } u(Q) = \text{diam } (t \circ f)(Q) = \text{diam } t(P) = R(g)$.

Lemma 7. Suppose f is a map from S^1 onto S^1, $\deg f = 1$, e is a number, $0 < e < \frac{1}{2}$, and $R(f) > 2 - e$. Then there is a map g from S^1 onto S^1 such that $\deg g = 1$, $R(g) \geq 2$, and $f \underset{e}{=} g$.

Proof. If $R(f) \geq 2$, then let $g = f$. Suppose $R(f) < 2$. Since $R(f) > \deg f$, let $[a,b]$ be a defining arc for $R(f)$. Let t be a lift of $f|[a,b]$, and let u be a lift of $f|[b,a]$ such that $u(b) = t(b)$. Then $f|[a,b] = \phi \circ t$ and $f|[b,a] = \phi \circ u$. By Lemma 4, $t([a,b]) = [t(a),t(b)]$, and $u([b,a]) = [t(a) + 1,t(b)]$. Let v be a linear map from $[t(a),t(b)]$ onto $[t(a),t(a) + 2]$, with $v(t(a)) = t(a)$, $v(t(b)) = t(a) + 2$. Let w be a linear map from $[t(a) + 1,t(b)]$ onto $[t(a) + 1, t(a) + 2]$, with $w(t(a) + 1) = t(a) + 1$, and $w(t(b)) = t(a) + 2$. Let g_1 be a function from $[a,b]$ into S^1, $g_1 = \phi \circ v \circ t$; let g_2 be a function from $[b,a]$ into S^1, $g_2 = \phi \circ w \circ u$. Let $g = g_1 \cup g_2$. Then g_1 and g_2 are continuous, and $g_1(a) = g_2(a)$, $g_1(b) = g_2(b)$, hence g is continuous. Since $v \circ t \underset{e}{=} t$, $w \circ u \underset{e}{=} u$, and ϕ is distance-preserving for intervals of length less than $\frac{1}{2}$, we have $f \underset{e}{=} g$. Since $\deg f = 1$ and $f \underset{\frac{1}{2}}{=} g$, $\deg g = 1$, and
$R(g) \geq \text{diam } v(t([a,b])) = 2$.

Theorem 3. If M is a p.n.c.c.l. continuum, and M is in class 2, then M is homeomorphic to $\varprojlim (Y_i, g_i^{i+1})$ such that each Y_i is S^1, $\deg g_i^{i+1} = 1$, and $R(g_i^j) \geq 2$, for each pair of positive integers i and j, $i < j$.

Proof. Let $M = \varprojlim (X_i, f_i^{i+1})$, each $X_i = S^1$, and M is in class 2. Let e be the number sequence $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$. Let $p_1 = 1$. Let p_2 be the first positive integer j such that $R(f_1^j) > 2 - \frac{1}{2}$. Let $F_1^2 = f_{p_1}^{p_2}$. Let g_1^2 be a map from S^1 onto S^1 such that $g_1^2 \underset{\frac{1}{2}}{=} F_1^2$, and $R(g_1^2) \geq 2$.

We proceed by induction. Suppose $p_1, p_2, \ldots, p_n, p_{n+1}$ are defined; $F_1^2, F_2^3, \ldots, F_n^{n+1}$ are defined, with $F_i^{i+1} = f_{p_i}^{p_{i+1}}$ for $1 \leq i \leq n$; $g_1^2, g_2^3, \ldots, g_n^{n+1}$ are defined, with $R(g_i^{i+1}) \geq 2$, for $1 \leq i \leq n$; for each triple (k,i,j) of positive integers with $k \leq i < j \leq n + 1$,

$$g_k^i \circ F_i^j \underset{(1 - \frac{1}{2^{j-i}})e_i}{=} g_k^j$$

and

$$F_k^i \circ g_i^j \underset{(1 - \frac{1}{2^{j-i}})e_i}{=} F_k^j .$$

Using the uniform continuity of the maps from S^1 into S^1, let $a > 0$ such that if x and y are points of S^1 and $|x - y| < a$, then for $1 \leq k \leq i < j \leq n + 1$,

$$|g_k^i \circ F_i^j(x) - g_k^i \circ F_i^j(y)| < \frac{e_i}{2^{n+2-i}}$$

and

$$|F_k^i \circ g_i^j(x) - F_k^i \circ g_i^j(y)| < \frac{e_i}{2^{n+2-i}} .$$

Let $d = \min(a, \frac{1}{2}e_n)$. Let p_{n+2} be the first positive integer j such that $R(f_{p_{n+1}}^j) > 2 - \frac{1}{2}d$. Let $F_{n+1}^{n+2} = f_{p_{n+1}}^{p_{n+2}}$. Let g_{n+1}^{n+2} be a map from S^1 onto S^1 such that $g_{n+1}^{n+2} \underset{\frac{1}{2}d}{=} F_{n+1}^{n+2}$ and $R(g_{n+1}^{n+2}) \geq 2$. Let x be a point of S^1. Suppose $1 \leq k \leq i < n + 1$. Then

$$(*) \quad |g_k^i \circ F_i^{n+1}(F_{n+1}^{n+2}(x)) - g_k^i \circ F_i^{n+1}(g_{n+1}^{n+2}(x))| < \frac{e_i}{2^{n+2-i}} .$$

Also, $$|g_k^i \circ F_i^{n+1}(g_{n+1}^{n+2}(x)) - g_k^{n+1}(g_{n+1}^{n+2}(x))| < (1 - \frac{1}{2^{n+1-i}})e_i .$$

Hence $$|g_k^i \circ F_i^{n+1}(F_{n+1}^{n+2}(x)) - g_k^{n+1}(g_{n+1}^{n+2}(x))| < (1 - \frac{1}{2^{n+2-i}})e_i .$$

The last inequality implies

$$g_k^i \circ F_i^{n+2} \underset{(1 - \frac{1}{2^{n+2-i}})e_i}{=} g_k^{n+2} .$$

Similarly,

$$F_k^i \circ g_i^{n+2} \underset{(1 - \frac{1}{2^{n+2-i}})e_i}{=} F_k^{n+2} .$$

Now, since $\frac{1}{2}d \leq \frac{1}{2}e_{n+1}$,

$$g_{n+1}^{n+2} \underset{\frac{1}{2}e_{n+1}}{=} F_{n+1}^{n+2} .$$

Also, since $\dfrac{e_k}{2^{n+2-k}} = \dfrac{e_{n+1}}{2}$, the inequality (*) yields

$$F_k^{n+1} \circ g_{n+1}^{n+2} \underset{\frac{1}{2}e_{n+1}}{=} F_k^{n+2} .$$

Similarly,

$$g_k^{n+1} \circ F_{n+1}^{n+2} \underset{\frac{1}{2}e_{n+1}}{=} g_k^{n+2} .$$

Thus, for each triple (k,i,j) of positive integers, with $k \leq i < j \leq n+2$,

$$g_k^i \circ F_i^j \underset{(1 - \frac{1}{2^{j-i}})e_i}{=} g_k^j$$

and

$$F_k^i \circ g_i^j \underset{(1 - \frac{1}{2^{j-i}})e_i}{=} F_k^j .$$

Recursively, there exists a sequence $(F_1^2, F_2^3, F_3^4, \ldots)$ of maps, a sequence $(g_1^2, g_2^3, g_3^4, \ldots)$ of maps, with $R(g_i^{i+1}) \geq 2$, and a decreasing sequence e of positive numbers with sequential limit 0, such that for each triple (k,i,j) of positive integers, with $k \leq i < j$,

$$g_k^i \circ F_i^j \underset{e_i}{=} g_k^j$$

and $$F_k^i \circ g_i^j \underset{e_i}{=} F_k^j$$

Let $K = \underset{\leftarrow}{\text{Lim}}\ (X_i, g_i^{i+1})$. By Theorem C, K is homeomorphic to $\underset{\leftarrow}{\text{Lim}}\ (X_i, F_i^{i+1})$, which is homeomorphic to M by Theorem A. Since $R(g_i^{i+1}) \geq 2$, for each i, Lemma 6 yields $R(g_i^j) \geq 2$ for $i < j$. This completes the proof.

A similar pattern of argument yields

Theorem 4. If M is a p.n.c.c.l. continuum in class B, then M is homeomorphic to $\underset{\leftarrow}{\text{Lim}}\ (Y_i, g_i^{i+1})$ such that each $Y_i = S^1$, $\deg g_i^{i+1} = 1$, and $R(g_i^j) \geq 3$ for each pair of positive integers i and j, with $i < j$.

D.R. Read proved in [7, Theorem 10] that each map from a continuum onto an arc is weakly confluent.

Lemma 8. Suppose each of f and g is a map from S^1 onto S^1; $R(f) > \deg f \geq 1$; $R(g) > \deg g \geq 1$; $R(f) \geq 2$. Then $R(g \circ f) \geq ([R(f)] - 2)\deg g + R(g)$, in which $[R(f)]$ is the greatest integer not exceeding $R(f)$.

Proof. Let [a,b] be a defining arc for R(f), [c,d] a defining arc for R(g), v a lift for $f|[a,b]$, u a lift for $g|[c,d]$. Let $\deg g = n$. Let t be a map from S^1 into the numbers such that $g = I^n.(\phi \circ t)$. The interval $[v(a),v(b)]$ is contractible with respect to S^1 (c.r.S^1), thus $g \circ \phi|[v(a),v(b)]$ is inessential. Let z be a lift of $g \circ \phi|[v(a),v(b)]$. Letting h be the identity map on the real line, we have

$g \circ \phi|[v(a),v(b)] = \{I^n.(\phi \circ t)\} \circ \phi|[v(a),v(b)]$

$= (I^n \circ \phi).(\phi \circ t \circ \phi)|[v(a),v(b)] = (\phi \circ (nh)).(\phi \circ t \circ \phi)|[v(a),v(b)]$

$= \phi \circ (nh + t \circ \phi)|[v(a),v(b)] = \phi \circ z.$

Consider the arcs [a,b] and [v(a),v(b)]. Let c' be the least number x, $v(a) \leq x$, such that $\phi(x) = c$, and d' be the greatest number y, $y \leq v(b)$, such that $\phi(y) = d$. Let c" be the greatest number x, $x < d'$, such that $\phi(x) = c$. Then $c'' - c'$ is an integer, and $c'' - c'' \geq [R(f)]-2$. Since v is weakly confluent, let $[a',b']$ be an arc in $[a,b]$ such that $v([a',b']) = [c',d']$.

Let w be a lift of $g \circ f|[a',b']$. Then $\phi \circ w = g \circ f|[a',b']$ $= g \circ \phi \circ v|[a',b'] = \phi \circ z \circ v|[a',b']$. We have $z(c'') - z(c') = nh(c'') + t(\phi(c'')) - nh(c') - t(\phi(c')) = n(c'' - c')$. Also $g \circ \phi|[c'',d'] = g|[c,d] \circ \phi|[c'',d'] = \phi \circ u \circ \phi|[c'',d'] = \phi \circ z|[c'',d']$. Thus $z(d') - z(c'') = u(d) - u(c) = R(g)$. Hence

$\operatorname{diam} w([a',b']) = \operatorname{diam} z([c',d']) \geq z(d') - z(c')$

$= z(d') - z(c'') + z(c'') - z(c') = R(g) + n(c'' - c')$. We have $R(g \circ f) \geq R(g) + ([R(f)] - 2)\deg g$, completing the proof.

Corollary. Suppose M is a p.n.c.c.l. continuum, $M = \varprojlim (S^1, f_i^{i+1})$, such that for each i, $\deg f_i^{i+1} = 1$ and $R(f_i^{i+1}) \geq 3$. Then for each positive integer j, the sequence $(R(f_j^{j+1}), R(f_j^{j+2}), R(f_j^{j+3}), \ldots)$ increases without bound.

CHAPTER III

MAPPING CHAINABLE CONTINUA ONTO CIRCLE-LIKE CONTINUA

In [8], Henderson proved that no non-planar circle-like continuum is the continuous image of a continuum c.r.S^1. In [4], Rogers proved that no chainable continuum can be mapped onto a circle-like continuum which is "self-entwined" (in his sense). In this chapter, Henderson's result is extended to include the circle-like continua which are self-entwined (in my sense). Also, two theorems are proved, each of which states necessary and sufficient conditions for a circle-like continuum to be the continuous image of a chainable continuum.

Lemma 9. If X is a continuum, and f a map from X onto S^1, and A an arc in S^1, and B the complementary arc of A, then either there is a subcontinuum H of X such that f(H) = A or there is a subcontinuum K of X such that f(K) = B.

Proof. Let A be an arc in S^1, and B its complement. One easily sees that the proposition holds in case X is an interval of length at least 1, and f is ϕ. Suppose X is a continuum. If f is an essential map from X onto S^1 then f is weakly confluent, and we have the conclusion of the lemma. If f is an inessential map from X onto S^1 then any lift map t of f is weakly confluent; letting D be the appropriate subinterval of t(X), and M a subcontinuum of X such that t(M) = D, either

$$f(M) = \phi(t(M)) = \phi(D) = A$$

or $$f(M) = \phi(t(M)) = \phi(D) = B.$$

Theorem 5. Suppose M is a self-entwined p.n.c.c.l. continuum. Then M is not the continuous image of a continuum c.r.S^1.

Proof. Suppose M is self-entwined, and X is a continuum c.r.S^1. We may assume, by Theorem 4, that $M = \varprojlim (S^1, f_i^{i+1})$, with deg $f_i^{i+1} = 1$, and $R(f_i^{i+1}) \geq 3$, for each i. Suppose g is a map from X onto M. Then $PR_1 \circ g$ is inessential; let u be a lift of $PR_1 \circ g$. Let, by the corollary to Lemma 8, n be a positive integer such that $R(f_1^n) > (\text{diam } u(X)) - 1$. Let $[a,b]$ be a defining arc for $R(f_1^n)$; t a lift of $f_1^n|[a,b]$; v a lift of $f_1^n|[b,a]$. Suppose H is a subcontinuum of X such that $PR_n \circ g(H) = [b,a]$. Then $\phi \circ u|H = PR_1 \circ g|H = f_1^n \circ PR_n \circ g|H = \phi \circ v \circ PR_n \circ g|H$. By Lemma 4, diam $u(H) =$ diam $v(PR_n(g(H))) =$ diam $v([b,a]) = R(f_1^n) - 1 >$ diam $u(X)$, a contradiction. Similarly, if K is a subcontinuum of X such that $PR_n \circ g(K) = [a,b]$, then diam $u(K) =$ diam $t([a,b]) = R(f_1^n) >$ diam $u(X)$, a contradiction.

Theorem 6. A circle-like continuum is the continuous image of a chainable continuum if and only if it is the continuous image of a continuum c.r.S^1.

Proof. Necessity is trivial, since chainable continua are c.r.S^1. Suppose $C = \varprojlim (S^1, f_i^{i+1})$, and g is a map from a continuum X onto C, with X c.r.S^1. Let i be a positive integer. Let t_i and t_{i+1} be lifts of $PR_i \circ g$ and $PR_{i+1} \circ g$, respectively. Since the arc is c.r.S^1, let $f_i^{i+1} \circ \phi|t_{i+1}(X) = \phi \circ h$. We have

$$PR_i \circ g = f_i^{i+1} \circ PR_{i+1} \circ g$$

$$\phi \circ t_i = f_i^{i+1} \circ \phi \circ t_{i+1}$$

$$\phi \circ t_i = \phi \circ h \circ t_{i+1} .$$

Since X is connected, let M be an integer such that $t_i = h \circ t_{i+1} + M$.

Let h' be the map h + M. Then $t_i = h' \circ t_{i+1}$. We have h' a map from $t_{i+1}(X)$ onto $t_i(X)$, and $\phi \circ h' = \phi \circ h = f_i^{i+1} \circ \phi | t_{i+1}(X)$.

Let, for each positive integer j, $Y_j = t_j(X)$, with t_j a lift of $PR_j \circ g$; $p_j = \phi | Y_j$; k_j^{j+1} be the map from Y_{j+1} onto Y_j such that $p_j \circ k_j^{j+1} = f_j^{j+1} \circ p_{j+1}$. Then, by Theorem B, C is the continuous image of $\varprojlim (Y_i, k_i^{i+1})$, a chainable continuum.

To prove Theorem 7, the main result, a technical lemma is required.

Lemma 10. Suppose each of f and g is a map from S^1 onto S^1 such that deg f = deg g = 1, $R(g) \geq 2$, and d is a number, $0 \leq d < 1$, such that $R(f \circ g) \leq 2 + d$ and $R(f) \leq 2 + d$. Let [a,b] be a defining arc for R(g) and w be a lift of $g|[a,b]$. Let $w([a,b]) = [p-1,q]$. The map $f \circ \phi | [p,p+1]$ is inessential; let t be a lift of it. Then diam $t([p,p+1]) \leq 1 + d$.

Proof. Let $f \circ \phi | [p,p+1] = \phi \circ t$. Let $t([p,p+1]) = [A,B]$. Suppose $B - A > 1 + d$. Let $p < x < p+1$. Then $\phi([p,x])$ is an arc. Let z be a lift of $f|[\phi(p),\phi(x)]$. Then $f \circ \phi|[p,x] = \phi \circ t|[p,x] = \phi \circ z \circ \phi|[p,x]$. We may assume that $z \circ \phi|[p,x] = t|[p,x]$.

Since this argument holds for each number x between p and p+1, there is a map u on the ray $[\phi(p),\phi(p+1))$ such that $u \circ \phi|[p,p+1) = t|[p,p+1)$. Now, if $p < x < p+1$, $f|[\phi(p),\phi(x)] = \phi \circ u|[\phi(p),\phi(x)]$. By Lemma 2, since deg f = 1, $\lim_{x \to p+1} u(\phi(x)) = u(\phi(p)) + 1$.

There is a proper subinterval Y of [p,p+1] such that $t(Y) = [A,B]$. For: We have $t([p,p+1))$ connected, and $t([p,p+1]) = t([p,p+1)) \cup t(\{p+1\})$. Hence one of 3 statements is true:

(a) $t([p,p+1)) = (A,B]$; (b) $t([p,p+1)) = [A,B)$; (c) $t([p,p+1)) = [A,B]$.

Suppose (a) holds. Then $t(p+1) = A$. But

$$t(p+1) = \underset{x \to p+1}{\text{Limit}}\, t(x) = \underset{x \to p+1}{\text{Limit}}\, u(\phi(x)) = u(\phi(p)) + 1.$$ Since $u(\phi(p)) \geq A$,

$t(p+1) \geq A+1$, a contradiction. Suppose (b) holds. Then $t(p+1) = B$. As before, $t(p+1) = u(\phi(p)) + 1 = t(p) + 1$. Thus $t(p) = t(p+1) - 1 = B - 1 > A$, since $B - A > 1$. Hence there is a number e, $p < e < p+1$, such that $t(e) = A$. Then $t([e,p+1]) = [A,B]$. Suppose (c) holds. Then there are numbers j and k in $[p,p+1)$ such that $t(j) = A$ and $t(k) = B$.

In either case, there is a proper subinterval $[e,r]$ of $[p,p+1]$ such that $t([e,r]) = [A,B]$. We may assume that the endpoints of $[e,r]$ are mapped to the endpoints of $[A,B]$ by t. In case $r < p+1$, $t|[e,r] = u \circ \phi|[e,r]$. In case $r = p+1$, let u' be a lift of $f|[\phi(e),\phi(r)]$ such that $u'(\phi(e)) = t(e)$. Then, by the previous argument, $t|[e,r] = u' \circ \phi|[e,r]$. Relabel $u = u'$ if necessary. Then either $u(\phi(e)) = A$ and $u(\phi(r)) = B$ or $u(\phi(e)) = B$ and $u(\phi(r)) = A$.

Suppose $u(\phi(e)) = B$. Let v be a map from the ray $[\phi(e),\phi(e))$ into the numbers, v an extension of u, such that $f|[\phi(e),\phi(e)) = \phi \circ v$. By Lemma 2, $\underset{x \to e+1}{\text{Limit}}\, v(\phi(x)) = v(\phi(e)) + 1 = B + 1$. But $B + 1 - A > 2 + d$,

and $v(\phi(r)) = u(\phi(r)) = A$. Hence there is a point y of S^1, $\phi(r) < y < \phi(e)$, such that $v(y) - v(\phi(r)) = v(y) - A > 2 + d$, contradicting $R(f) \leq 2 + d$. Therefore $u(\phi(e)) = A$ and $u(\phi(r)) = B$.

Now, $[e-1,r] \subseteq [p-1,p+1] \subseteq w([a,b])$. By an argument similar to that for Lemma 8, there is an arc $[a',b']$ lying in $[a,b]$, such that $\deg([a',b'],f \circ g) \geq 1 + B - A > 2 + d$, a contradiction. This completes the proof.

The following lemma is easily verified.

Lemma 11. If u is a map from a continuum A onto a continuum B, and v is a map from B onto a continuum C, and v o u is weakly confluent, then v is weakly confluent.

Definition. By "class W" we shall mean the class of all continua Y such that if X if a continuum, and f a map from X onto Y, then f is weakly confluent.

Theorems 10 and 11 of [7] assert that arcs and arc-like continua are in class W.

Theorem 7. If C is a circle-like continuum then C is the continuous image of a chainable continuum if and only if either C is chainable or C is not in class W.

Proof. Suppose C is a circle-like continuum not in class W. Let $C = \varprojlim (S^1, f_i^{i+1})$, and let g be a non-weakly confluent map from a continuum X onto C. Suppose that for all but finitely many positive integers i, PR_i o g is essential. Then for almost all i, PR_i o g is weakly confluent. The argument for [7, Theorem 11] implies that g is weakly confluent, a contradiction. Hence for infinitely many, and therefore all, positive integers i, PR_i o g is inessential. By an argument similar to that for Theorem 6, C is the continuous image of a chainable continuum.

Suppose that C is the continuous image of a chainable continuum X under the map g, and C is not chainable. By [8], C is planar, and by Theorem 5, C is not self-entwined. Let $C = \varprojlim (S^1, f_i^{i+1})$, with $\deg f_i^{i+1} = 1$ for each i. Let, for each positive integer j, t_j be a lift of PR_j o g. Now, there exist a sequence $(d_1, d_2, \ldots)$ of numbers, with

$0 \le d_i < 1$ for each i, and a sequence $(V_1, V_2, \dots)$ of intervals, with $V_i \subseteq t_i(X)$ and diam $V_i = 1$ for each i, such that if i and j are positive integers with $i < j$, and p is a lift of $f_i^j \circ \phi|V_j$, then diam $p(V_j) \le 1+d_i$. The proof of this assertion involves two cases.

Case 1. Suppose C is decomposable. By Theorem 2, C is homeomorphic to a member of class 1. Let, for each positive integer i, d_i be a number, $0 \le d_i < 1$, such that for $k > i$, $R(f_i^k) \le 1 + d_i$. Let, for each positive integer p, V_p be any subinterval of $t_p(X)$ with length 1. Suppose i and j are positive integers, $i < j$. For any proper subinterval U of V_j, $\phi(U)$ is an arc in S^1, and $\deg(\phi(U), f_i^j) \le R(f_i^j) \le 1 + d_i$; thus if p is a lift of $f_i^j \circ \phi|V_j$, diam $p(U) \le 1 + d_i$. Since this holds for each such U, diam $p(V_j) \le 1 + d_i$.

Case 2. Suppose C is indecomposable. By Theorems 2 and 3, we may assume that for each i and j, $i < j$, $R(f_i^j) \ge 2$. Since C is not self-entwined, let, for each i, d_i be a number, $0 \le d_i < 1$, such that for $k > i$, $R(f_i^k) \le 2 + d_i$. Suppose j is a positive integer. By an argument similar to that for Theorem 6, let u be a lift of $f_j^{j+1} \circ \phi|t_{j+1}(X)$ such that $u(t_{j+1}(X)) = t_j(X)$. Let $[a,b]$ be a defining arc for $R(f_j^{j+1})$. Let A be the least number in $\phi^{-1}(a) \cap t_{j+1}(X)$, and let B be the least number in $\phi^{-1}(b) \cap t_{j+1}(X)$. Let r be a lift of $f_j^{j+1}|[a,b]$ such that $r(b) = r(\phi(B)) = u(B)$. Let y be a lift of $f_j^{j+1}|[b,a]$ such that $y(b) = r(b)$. If $A < B$, then $u([A,B]) = r([a,b]) = [r(a),r(b)]$, and $[r(a)+1,r(a)+2] \subseteq t_j(X)$. If $B < A$, then $u([B,A]) = y([b,a]) = [r(a)+1,r(b)]$ by Lemma 4, and $[r(a)+1,r(a)+2] \subseteq t_j(X)$. Let $V_j = [r(a)+1,r(a)+2]$.

Suppose i and j are positive integers, $i < j$. By Lemma 10, if p is a lift of $f_i^j \circ \phi|V_j$, then diam $p(V_j) \leq 1 + d_i$.

Let $(d_1, d_2, \ldots)$ be a sequence of numbers and $(V_1, V_2, \ldots)$ a sequence of intervals as described. Since each map t_i is weakly confluent, let, for each positive integer j, K_j be a subcontinuum of X such that $t_j(K_j) = V_j$. Let $(K_{i_1}, K_{i_2}, K_{i_3}, \ldots)$ be a subsequence of K with a sequential limiting set M. Then M is a continuum.

Now, $g(M) = C$. For: Let y be an element of C. Since, for each j, $PR_j \circ g(K_j) = \phi \circ t_j(K_j) = \phi(V_j) = S^1$, let, for each n, x_n be a point of K_{i_n} with $PR_{i_n} \circ g(x_n) = y_{i_n}$. Let z be a cluster point of x, z in M. Suppose $g(z) \neq y$. Let n be a positive integer such that $PR_{i_n} \circ g(z) \neq y_{i_n}$. Let U and D be disjoint open sets in S^1 such that $PR_{i_n}(g(z))$ is in U and y_{i_n} is in D. Let $Q = (PR_{i_n} \circ g)^{-1}(U)$. Then Q is open in X, and z is in Q. Hence there exists $m > n$ with x_m in Q. Therefore $y_{i_n} = f_{i_n}^{i_m}(y_{i_m}) = f_{i_n}^{i_m}(PR_{i_m}(g(x_m)) = PR_{i_n} \circ g(x_m)$ which is in U, since x_m is in Q. This involves a contradiction.

Now, for each j, diam $t_j(M) \leq 1 + d_j$. For: Suppose n is a positive integer such that diam $t_n(M) > 1 + d_n$. Let $t_n(M) = [p,q]$. Let p' and q' be points of M such that $t_n(p') = p$, and $t_n(q') = q$. Let a be a number such that $0 < a < \frac{1}{2}(q - p - 1 - d_n)$. Let b be a positive number such that if z is a point of X, with $\text{dist}_X(p',z) < b$, then $|t_n(p') - t_n(z)| < a$, and if z is a point of X, with $\text{dist}_X(q',z) < b$, then $|t_n(q') - t_n(z)| < a$. Let m be an integer, $m \geq n$, such that if $j \geq m$, then there are points x_j and y_j in K_{i_j} such that $\text{dist}_X(p',x_j) < b$ and $\text{dist}_X(q',y_j) < b$.

Consider $t_{i_m}(K_{i_m}) = V_{i_m}$. Let u be a lift of $f_n^{i_m} \circ \phi|V_{i_m}$. Then

$\phi \circ t_n|K_{i_m} = PR_n \circ g|K_{i_m} = f_n^{i_m} \circ PR_{i_m} \circ g|K_{i_m} = f_n^{i_m} \circ \phi \circ t_{i_m}|K_{i_m}$

$= \phi \circ u \circ t_{i_m}|K_{i_m}$. Hence $\operatorname{diam} t_n(K_{i_m}) = \operatorname{diam} u(t_{i_m}(K_{i_m})) \leq 1 + d_n$.
Let x_m and y_m be points of K_{i_m} such that $\operatorname{dist}_X(p',x_m) < b$ and $\operatorname{dist}_X(q',y_m) < b$. Then $|p - t_n(x_m)| = |t_n(p') - t_n(x_m)| < a$ and $|q - t_n(y_m)| < a$. We have

$$|t_n(x_m) - t_n(y_m)| \geq (q - p) - |p - t_n(x_m)| - |q - t_n(y_m)|$$
$$> q - p - 2a > 1 + d_n.$$

Thus $\operatorname{diam} t_n(K_{i_m}) > 1 + d_n$, a contradiction.

Suppose j is a positive integer. Then $1 \leq \operatorname{diam} t_j(M) \leq 1 + d_j < 2$, and $\phi|t_j(M)$ is not weakly confluent. Hence $PR_j \circ g|M = \phi \circ t_j|M$ is not weakly confluent by Lemma 11. Since $\deg f_i^{i+1} = 1$ for each i, PR_j is an essential map from C onto S^1, thus PR_j is weakly confluent. If $g|M$ were weakly confluent, then $PR_j \circ g|M$ would be weakly confluent. Therefore $g(M)$ is C and $g|M$ is not weakly confluent, implying that C is not in class W. This completes the proof.

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