NUMERICAL SOLUTIONS OF FREE CONVECTION FROM AN ISOTHERMAL INCLINED PLATE

A Thesis

Presented to

The Faculty of the Department of Mechanical Engineering

University of Houston

In Partial Fulfillment

of the Requirments for the Degree

Master of Science in Mechanical Engineering

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by

Chung-Jen Kau

August, 1967

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ABSTRACT

The differential equations decribing the free convection heat transfer about an inclined plate have been solved by a numerical integration method. The partial differential equations were transformed into a coupled set of ordinary nonlinear differential equations via a similarity transform. These nonlinear differential equations are subject to boundary conditions at the origin and at infinity. A suitable approximation of infinity was made and the solution was obtained via a quasilinearization procedure.

The following conclusions are drawn :

(1) The thickness of both the hydrodynamic and thermal boundary layers increases with the inclination to the vertical. The maximum velocity within the boundary layer decreases with the increasing inclination.

(2) For small and medium inclinations, there is good agreement between the calculated results and experiment results.

(3) Boundary layer assumptions, implying that the distance along the plate is much larger than the boundary layer thickness, are valid for the cases of small inclination.

(4) A modified empirical Nusselt number(for the vertical plate) was found having excellent correlation with numeri-

cal results presented for the cases Prandtl number less than 100. For larger Prandtl number cases, the modified Eckert equation has better correlation.

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CHAPTER I

INTRODUCTION

Free convection is the phenomena of heat and mass transfer resulting from a fluid located in a force field in the presence of a temperature gradient. The density variation with temperature causes an unbalance of the body forces resulting in fluid motion. This presentation is concerned with laminar free convection from an isothermal inclined flat surface.

The most common model previously studied is the vertical plate immersed in an initially stationary fluid. Pohlhausen (1) developed an exact solution for a laminar viscous fluid. Schmidt and Beckmann (2) experimentally established the boundary layer characteristics which agree with the analytical preditions of Pohlhausen (see reference (1)). Ostrach (3) analytically investigated the vertical flat plate for large Grashof numbers. The analysis compares favorably with Schmidt and Beckmann's temperature distribution but varies significantly from the measured velocity distributions. Albers numerically solved the vertical plate case (see Appendix B of reference (3)) by transforming the ordinary differential equations and estimating their eigenvalues. He then numerically integrated the equations. Improved estimates of the eigenvalues were then made on the basis of the preceding runs and the process was repeated successively until a satisfactory solution was obtained. The numerical results he obtained were for the cases of Prandtl numbers ranging from 0.01 to 1000. Sparrow and Gregg have also made signigicant contributions in this field. They investigated the cases of uniform surface heat flux (4) and the problem of variable fluid-property free convection (5) about a vertical flat plate. The effect of other types of acceleration fields on the free convection flow about a vertical flat plate has also been studied, such as Emery's investigation (6) on the effect of a magnetic field and Lemlich's study on the problem of spatially varying acceleration about an isothermal flat plate (7).

The case of the inclined plate in a gravitational field has received little attention. An experimental study of heat transfer from an isothermal inclined flat plate was made by Rich (8). He measured the heat transfer coefficients and the temperature distributions for the case of air at angles of inclination ranging from 0 to 40 degrees measured from the vertical position and with the temperature differences from the plate surface and the surrounding of 200°F to 260°F. Rich concluded that the Nusselt number for the inclined plate can be predicted from the vertical plate by multiplying the latter a factor of $(\cos\Phi)^{1/4}$, where Φ is the angle of inclination from the vertical position. Prior to Rich's experiments, Tautz (9) had also measured the heat transfer from a inclined square plate. According to Tautz's experiments, the combined (upper and lower surfaces) heat transfer at angles greater than 45° is constant and equal to that from a vertical; at angles smaller than 45° the heat transfer can be obtained by linear interpolation between values for the horizontal and vertical positions.

In 1963, Michiyoshi (10) proposed third-degree polynomial velocity and temperature profiles for the case of free convection flow from an inclined flat plate. Approximate solutions were then obtained by making use of the Karman-Pohlhausen integral method (11). The problem of inclined flat plate has also been investigated analytically by means of perturbation to the boundary equations by Guinle (12).

In this presentation, a numerical study of free convection flow as well as heat transfer from an isothermal inclined plate will be made by using finite difference approximations and quasilinearization techniques for the solution of the governing differential equations.

CHAPTER II

DEVELOPMENT OF EQUATIONS

Since all of the natural convection processes involve density variations, any strict analyical study of natural convection problems should include density variations. However, most of the previous studies of natural heat convection model have neglected it. Such a simplification does not appear unreasonable for gases as well as some other commonly involved fluids. This intuitive feeling has corrorated in a formal manner by Ostrach in his paper (3). In many cases of engineering applications, especially in many processes of practical importance where the temperature difference between fluid and plate is small, the error incurred by the assumption is very small. Nevertheless, the effect of the density variations which caused by a nonuniform temperature field will be retained for the case presented.

The equations of motion for convection are generated from Navier-Stokes equations. For a two dimensional, steady state, laminar free convection flow, the general conservation equations are:

Momentum in x-direction:

$$\left[u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \right] \rho = -\frac{\partial P}{\partial x} - \rho g \cos\phi + \mu \nabla^2 u$$

Momentum in y-direction:

$$\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]\rho = -\frac{\partial p}{\partial y}-\rho g \sin\phi+\mu\nabla^2 v$$

Energy:

$$\left[u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}\right] = \alpha\nabla^2 t \qquad (2\text{-}1)$$

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0$$

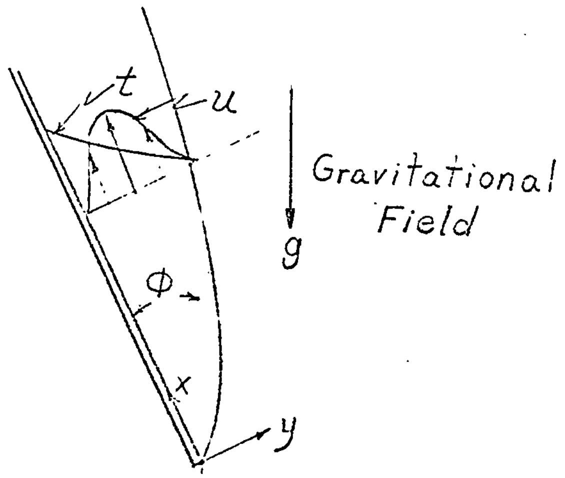

Figure 2-1

Inclined Plate System

with boundary conditions:

$$y=0 \;:\; u=0 \;;\; v=0 \;;\; t=t_w$$
$$y\rightarrow\infty \;:\; u\rightarrow 0 \;;\; v\rightarrow 0 \;;\; t=t_\infty \;;\; p\rightarrow p_\infty$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2}$$

μ =absolute viscosity, lb/ft-sec

α = thermal diffusivity, $k/Cp\,\rho$, ft^2/sec

ϕ = angle of inclination measured from the vertical

k = thermal conductivity, Btu/hr-ft^2-F^o

Cp = specific heat, Btu/lb-F^o

The physical model and the coordinate system are shown in Figure 2-1. The subscript " ∞ " indicates the properties at the outer edge of the boundary layer, of more precisely at infinity, and subscript " w " indicates those at the surface of the plate.

According to the concept of boundary layer theory, for the case of free convection the flow is confined to a very thin layer in the immediate neighborhood of the flat plate. The thickness of this boundary layer increases along the plate in the downstream direction; Figure 2-1 represents digrammatically the velocity distribution and temperature distribution in such a boundary layer. The dimensions across it are considerably exaggerated. The fluid is considered to be still in the region characterized by $x \leqslant 0$.

Since the boundary layer thickness is very small compare with any charateristic dimension of the plate, it can be expected that the change of the parameters like v, u and p characterizing the flow is much more rapid in the y direction than in the x direction. Thus an order of magnitude analysis (19) can be made for the simplification of the partial differential equations.

After making the analysis the system of equations is reduced to the form:

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\rho = -\frac{\partial p}{\partial x} - \rho g \cos\phi + \mu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = -\rho g \sin\phi \tag{2-2}$$

$$\left[u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y}\right] = \alpha\frac{\partial^2 t}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

with boundary conditions:

$$y = 0 \;:\; u = 0 \;;\; v = 0 \;;\; t = t_w$$

$$y \rightarrow \infty \;:\; u \rightarrow 0 \;;\; v \rightarrow 0 \;;\; t \rightarrow t_\infty \;;\; p \rightarrow p_\infty$$

First equation of equations (2-2) will be rearranged by adding to and substracting from the right-hand side the term $\rho_\infty g \cos\phi$. Thus, it becomes:

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\rho =$$

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} - \rho_\infty g \cos\phi - (\rho - \rho_\infty) g \cos\phi \tag{2-3}$$

In specifying the above equation at the outer edge of the boundary layer where u=0, v=0 and $\rho = \rho_\infty$ the following result is obtained:

$$\frac{\partial p}{\partial x} + \rho_\infty g \cos\phi = 0 \qquad (2\text{-}4)$$

When the flow velocity is small compared with the sonic velocity say, a Mach number less than 0.3, the fluid involved is always considered imcompressible. Thus, for the case of free convection, density variations will be assumed to be temperature dependent only. The coefficient of thermal expansion β is defined as:

$$\beta = \lim_{\Delta t \to 0} \frac{\Delta V / V_i}{\Delta t} \qquad (2\text{-}5)$$

where;

V_i = initial volume

ΔV and Δt are increments of volume and temperature respectively. For engineering purposes, an average can be defined as:

$$\beta = \frac{1}{V_i}\frac{\Delta V}{\Delta t} = \rho_i \frac{\frac{1}{\rho_f} - \frac{1}{\rho_i}}{t_f - t_i} = \frac{1}{\rho_f}\frac{\rho_i - \rho_f}{t_f - t_i} \qquad (2\text{-}6)$$

where the subscript "f" stands for the final state and "i" stands for the initial state.

Equation (2-6) can be expressed as :

$$\rho_i - \rho_f = \beta \rho_f (t_f - t_i) \qquad (2\text{-}7)$$

For the case here the above expression should be:

$$\rho - \rho_\infty = \beta \rho_\infty (t_\infty - t) \tag{2-8}$$

Thus the first equation of equation (2-2) can be re-written as:

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right] = \nu \frac{\partial^2 u}{\partial y^2} + \beta g \cos\phi (t - t_\infty) \tag{2-9}$$

where ;

ν = kinematic viscosity, μ/ρ , ft^2/sec

It is worth-while noting that for the case of a gas, Sparrow and Gregg (5) developed a reference temperature which was defined as :

$$t_r = t_w - 0.38(t_w - t_\infty) \tag{2-10}$$

To extend the constant property results to the variable-property situations, the suggestion was made to replace the thermal expansion coefficient β by $1/t_\infty$ and to evalute the other properties of the involved fluid at the defined reference temperature t_r .

From the second of equations (2-10), it is known that a pressure variation across the boundary layer (which is different from $\frac{\partial P}{\partial y} = 0$ for the case of vertical plate) does exist for the inclined plate. A more rigorous expression (see the discussion of reference (8)) should be:

$$\frac{\partial p}{\partial y} = \rho_\infty g \sin\phi \; \frac{t - t_\infty}{t_\infty} \tag{2-11}$$

Thus, the governing equations become:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial x^2} - \beta g \cos\phi \, (t - t_\infty)$$

$$\frac{\partial p}{\partial y} = \rho_\infty g \sin\phi \, \beta \, (t - t_\infty) \tag{2-12}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

with boundary conditions :

$$y = 0 \; : \; u = 0 \quad ; \quad v = 0 \quad ; \quad t = t_w$$

$$y \rightarrow \infty \; : \; u \rightarrow 0 \quad ; \quad v \rightarrow 0 \quad ; \quad t \rightarrow t_\infty \; ; \; p \rightarrow p_\infty$$

Quite a number of simplifications have been made to the orginal system of partial differential equations. However, there is still a fourth order system of equations to be solved. It is fortunate that the similarity transfromation which Pohlhausen (1) has applied to the governing partial differential equations of the case of the vertical plate can be suitably applied to this case. Furthermore, by examining the equations (2-10), it is found that the dependent variable, p, which appeared in the second equation of this system does not appear in any other equations of the system. This implies that the second equation can be treated separately in the numerical procedures.

In order to transform this system of partial differential equations, the solution of equations (2-10) will be written in terms of a stream funtion Ψ, defined by the relations :

$$u = \frac{\partial \Psi}{\partial y} \quad ; \quad v = -\frac{\partial \Psi}{\partial x} \qquad (2\text{-}13)$$

An independent variable η, the so-called similariry variable is also defined by:

$$\eta = c\,\frac{y}{\sqrt[4]{x}} \qquad (2\text{-}14)$$

where ;

$$c = \left[\frac{g(t_w - t_\infty)}{4\nu^2\, t_\infty}\right]^{\frac{1}{4}}$$

New dependent variables f and θ are given by :

$$f(\eta) = \left[\frac{\Psi}{x^{3/4}}\right]\left[\frac{1}{4\nu c}\right]$$

$$\theta(\eta) = \frac{(t - t_\infty)}{(t_w - t_\infty)} \qquad (2\text{-}16)$$

The velocity components now become :

$$u = 4\nu x^{1/2} c^2 f'(\eta)$$

$$v = \nu c x^{-1/4}[\eta f'(\eta) - 3 f(\eta)] \qquad (2\text{-}17)$$

The partial derivatives of the parameters which characterize the flow are related to the new variables as follows :

$$\frac{\partial \eta}{\partial x} = -\frac{1}{4}\, x^{-1}\eta$$

$$\frac{\partial \eta}{\partial y} = cx^{-1/4}$$

$$\frac{\partial u}{\partial x} = \nu c^2 x^{-1/2}(2f' - f''\eta)$$

$$\frac{\partial u}{\partial y} = 4\nu c^3 x^{-1/4} f''$$

$$\frac{\partial^2 u}{\partial y^2} = 4\nu^2 c^4 f''' \qquad (2\text{-}18)$$

$$\frac{\partial v}{\partial y} = \nu c^2 x^{1/2}(f''\eta - 2f')$$

$$\frac{\partial t}{\partial x} = -\frac{1}{4}x^{-1}\eta\,\theta'(t_\omega - t_\infty)$$

$$\frac{\partial t}{\partial y} = \theta' c x^{-1/4}(t_\omega - t_\infty)$$

$$\frac{\partial^2 t}{\partial y^2} = \theta'' c^2 x^{-1/2}(t_\omega - t_\infty)$$

where each prime represents the differentiation with respect to the new independent variable η.

Substituting all these partial derivatives into equations (2-12), two simultaneous nonlinear ordinary differential equations are obtained :

$$f''' + 3f''f - 2(f')^2 + \theta\cos\phi = 0$$

$$\theta'' + 3\,Pr\,f\,\theta' = 0 \qquad (2\text{-}19)$$

where ;

Pr = Prandtl number, $C_p\,\mu/k$, dimensionless

These equations are subject to the boundary conditions :

$$\eta = 0 \quad : \quad f = 0 \quad ; \quad f' = 0 \quad ; \quad \theta = 1$$

$$\eta \to \infty \quad : \quad f' \to 0 \quad ; \quad \theta \to 0$$

The pressure gradient normal to the surface of the plate is now written as :

$$\frac{\partial P}{\partial y} = \rho_\infty g \beta \theta \sin\phi \,(t_\omega - t_\infty) \tag{2-20}$$

with the boundary condition

$$y \to \infty \quad : \quad P \to P_\infty$$

It is observed that the pressure distribution normal to the surface can be obtained numerically by integrating the equation (2-20) together with equation (2-4) after equations (2-19) have been solved.

CHAPTER III

NUMERICAL PROCEDURES

The solution of a system of differential equations, subject to appropriate initial conditions, can generally be obtained by a forward integration method. The most common methods used for this integration are predictor-corrector methods and Runge-Kutta methods, because of their excellent stability and flexibility (13). For a system of equations including higher-order terms, a simple change of variable will usually change the system of equations into a system of first-order equations to facilitate programming.

For the boundary-value problems which do not provide a full set of initial conditions, the straight-forward application of numerical integration techniques used for initial value problems is often insufficient. If the problems are linear, superposition will yield a finite series of initial value problems whose weighted sum will satisfy the boundary conditions.

The governing simultaneous differential equations have already been derived for the case described in the preceding chapter are:

$$\begin{aligned} &f''' + 3ff'' - 2f'^2 + \theta\cos\phi = 0 \\ &\theta'' + 3Prf\theta' = 0 \end{aligned} \qquad (2\text{-}19)$$

with boundary conditions :

$$\eta = 0 : \quad f = 0 \quad ; \quad f' = 0 \quad ; \quad \theta = 1$$

$$\eta \to \infty : \quad f' \to 0 \quad ; \quad \theta \to 0$$

This is a fifth-order nonlinear boundary value problem. In order to obtain an equivalent first-order system of equations, equations (2-19) will be transformed by letting :

$$f = F_o$$

$$f' = F_o' = F_1$$

$$f'' = F_o'' = F_1' = F_2$$

$$f''' = F_o''' = F_1'' = F_2'$$

$$= -3f''f + 2f'^2 - \theta \cos\phi$$

$$\theta = F_3$$

$$\theta' = F_3' = F_4$$

$$\theta'' = F_3'' = F_4' = -3P_r f \theta'$$

Therefore, the first-order equations of the system are :

$$F_o' = F_1$$

$$F_1' = F_2$$

$$F_2' = -3F_2 F_o - 2F_1^2 - F_3 \cos\phi \tag{3-1}$$

$$F_3' = F_4$$

$$F_4' = -3P_r F_o F_3$$

with two-point boundary values :

$\eta = 0 \quad : \quad F_0 = 0 \ ; \ F_1 = 0 \ ; \ F_3 = 1$

$\eta \to \infty \quad : \quad F_1 \to 0 \ ; \ F_3 \to 0$

As a boundary value problem, the additional difficulties are caused by the non-linearity of the system. The quasilinearization techniques(14) will circumvent this problem. All terms in the system will be approximated by their Taylors expansions. For example, the nonlinear term $-3F_2F_0$ can be approximated by :

$$\begin{aligned} -3F_2F_0 \doteq & -3F_2^nF_0^n - F_0^n(F_2^{n+1} - F_2^n) \\ & -3F_2^n(F_0^{n+1} - F_0^n) \end{aligned} \qquad (3\text{-}2)$$

where subscript "n" and "n+1" indicate the nth and (n+1)st approximation. The (n+1)st approximation can be obtained after the calculation of the nth approximation.

Therefore, the linearized recurrence relationships of system are :

$$\begin{aligned} F_0'^{n+1} &= F_1^{n+1} \\ F_1'^{n+1} &= F_2^{n+1} \\ F_2'^{n+1} &= -3F_2^nF_0^n - 3F_0^n(F_2^{n+1} - F_2^n) \\ & \quad -3F_2^n(F_0^{n+1} - F_0^n) + 2(F_1^n)^2 \\ & \quad +4F_1^n(F_1^{n+1} - F_1^n) + F_3^{n+1}\cos\phi \\ F_3'^{n+1} &= F_4^{n+1} \end{aligned} \qquad (3\text{-}3)$$

$$F_4'^{n+1} = -3Pr\left[F_0^n F_3^n + F_0^n (F_3^{n+1} - F_3^n) + F_3^n (F_0^{n+1} - F_0^n) \right]$$

In matrix notation, equations (3-3) can be expressed in the following compact form :

$$\vec{F}'^{n+1} = [A] \times \vec{F}^{n+1} + \vec{K} \tag{3-4}$$

where ;

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -3F_0^n & 4F_1^n & -3F_0^n & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -3PrF_3^n & 0 & 0 & -3PrF_0^n & 0 \end{bmatrix} \tag{3-5}$$

and

$$\vec{F}^{n+1} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}^{n+1} ; \quad \vec{K}^n = \begin{bmatrix} 0 \\ 0 \\ 3F_2^n F_0^n - 2(F_1^n)^2 \\ 0 \\ -3PrF_3^n F_0^n \end{bmatrix}$$

The linearized equations can be solved by superposition. The general solution will be taken to have the form:

$$\vec{F} = \vec{F}_p^{(0)} + \sum_{j=1}^{5} \alpha_j \vec{F}_h^{(j)} \tag{3-6}$$

where the nonzero and zero superscripts on the right hand side represent homogeneous and particular solutions of equation (3-4) respectively. The superscript (j) indicates the jth linearly independeat homogeneous solution and α_j are the arbitrary constants picked to ensure that F^{n+1} satisfies the boundary conditions.

A particular solution of the first approximation can be obtained by integrating equation (3-4). Because of the incompleteness of the initial conditions, a set of initial conditions has to be assumed for starting the numerical integration. This is done by assuming any finite initial conditions.

In addition to the initial conditions, and approximation of the solution has to be used in the first iteration for starting the iteration. The initial approximation of the solutions in this case will be taken as:

$$\vec{F}(\eta) = \begin{bmatrix} (1-e^{-\eta})/2 \\ \sin(\frac{\pi}{10}\eta)e^{-\eta} \\ 0 \\ 1 - \eta/``\infty" \\ -1/``\infty" \end{bmatrix} \tag{3-7}$$

The first two elements of initial approximation (3-7) are approximated on the basis of the f profile and the non-dimensional velocity profile, f', of the solutions of the vertical plate (16). The third element is arbitrarily chosen. The last two elements are approximated by the interpolations of the two-point boundary conditions (see equations (3-3)) between $\eta = 0$ and $\eta = $ "∞" and their derivatives.

At this point it is necessary to look at the boundary conditions at infinity. Since the boundary layer thickness is assumed to be very small. The properties of the fluid at the infinite distance from the surface of the plate will be no difference from those at a certain finite distance outside the boundary layer. Thus, "∞" can be replaced by a well chosen finite value. This value may not be too large. For example, $\eta = 10$ or $\eta = 15$ may satisfactorily represent infinity. The discussion and illustration of the numerical techniques for this replacing of the boundary condition at infinity is found in reference (15).

Instead of assuming finite initial condition, a set of linearly independent initial conditions have to be assumed for the linearly independent homogeneous solutions. Usually, the columns of an appropriate (of same order as the system) non-singular matrix are assumed. Simply, and identity matrix may be used as the initial conditions :

$$\left[\vec{F}_h^{(1)}(0)\ \vec{F}_h^{(2)}(0)\ \vec{F}_h^{(3)}(0)\ \vec{F}_h^{(4)}(0)\ \vec{F}_h^{(5)}(0)\right] = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \tag{3-8}$$

Along with the initial approximation (3-7) five homogeneous solutions $\vec{F}_h^{(1)}$, $\vec{F}_h^{(2)}$, $\vec{F}_h^{(3)}$, $\vec{F}_h^{(4)}$ and $\vec{F}_h^{(5)}$ can be obtained by numerically integrate the homogeneous equation of equation (3-4), that is :

$$\dot{\vec{F}}(\eta) = [A] \times \vec{F} \tag{3-9}$$

The complete solution is then formed by adding to the particular solution the arbitrary multiplies of those five linearly independent homogeneous solutions, that is :

$$\vec{F} = \vec{F}_p + \sum_{j=1}^{5} \alpha_j \vec{F}_h^{(j)} \tag{3-10}$$

All of those solutions will be obtained as numerical data at discrete points of the range of integration. Five arbitrary constants have now to be determined to satisfy the boundary conditions. By specifying equation (3-10) at the points where the boundary conditions are given, it will result in five simultaneous equations :

At $\eta = 0$

$$F_0(0) = f(0) = 0 = \left[F_0(0)\right]_p + \sum_{j=1}^{5} \alpha_j \left[F_0^{(j)}(0)\right]_h$$

$$F_1(0) = f'(0) = 0 = \left[F_1(0)\right]_p + \sum_{j=1}^{5} \alpha_j \left[F_1^{(j)}(0)\right]_h$$

$$F_3(0) = \theta(0) = 1 = \left[F_3(0)\right]_p + \sum_{j=1}^{5} \alpha_j \left[F_3^{(j)}(0)\right]_h$$

and at η = "∞"

$$F_1(\infty) = f'(\infty) = 0 = \left[F_1(\infty)\right]_p + \sum_{j=1}^{5} \alpha_j \left[F_1^{(j)}(0)\right]_h$$

$$F_3(\infty) = \theta(\infty) = 0 = \left[F_3(\infty)\right]_p + \sum_{j=1}^{5} \alpha_j \left[F_3^{(j)}(0)\right]_h$$

or in matrix notation :

$$[B] \times \vec{\alpha} = \vec{Q} \tag{3-12}$$

where $[B]$ is the coefficient matrix

$$[B] = \begin{vmatrix} F_0^{(1)}(0) & F_0^{(2)}(0) & F_0^{(3)}(0) & F_0^{(4)}(0) & F_0^{(5)}(0) \\ F_1^{(1)}(0) & F_1^{(2)}(0) & F_1^{(3)}(0) & F_1^{(4)}(0) & F_1^{(5)}(0) \\ F_3^{(1)}(0) & F_3^{(2)}(0) & F_3^{(3)}(0) & F_3^{(4)}(0) & F_3^{(5)}(0) \\ F_1^{(1)}(\infty) & F_1^{(2)}(\infty) & F_1^{(3)}(\infty) & F_1^{(4)}(\infty) & F_1^{(5)}(\infty) \\ F_3^{(1)}(\infty) & F_3^{(2)}(\infty) & F_3^{(3)}(\infty) & F_3^{(4)}(\infty) & F_3^{(5)}(\infty) \end{vmatrix}$$

$$\vec{\alpha} = \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{vmatrix} \qquad ; \quad \vec{Q} = \begin{vmatrix} -[F_0(0)]_p \\ -[F_1(0)]_p \\ 1-[F_4(0)]_p \\ -[F_1(\infty)]_p \\ -[F_4(\infty)]_p \end{vmatrix}$$

Thus, the solution of equation (3-12) can be expressed in matrix notation as :

$$\vec{\alpha} = [B]^{-1} \times \vec{Q} \tag{3-13}$$

where $[B]^{-1}$ is the inverse of coefficient matrix $[B]$.

Numerically, in order to reduce the computational round-off error Gauss-Jordan maximum pivot method (17) will be employed to solve equation (3-12) for these arbitrary constants α_j .

As an improvement, all these computational procedures will be repeated again after replacing the assumed initial approximation with the results of preceding iteration. The iteration process will be terminated when the stable solution is reached. Accuracy of six significant figures may be obtained by terminating the iteration when the maximum absolute value of the arbitrary constants α_j is small, say, less than 10^{-6}.

For the case of air, the Prandtl number is about 0.72. Numerical solutions have been obtained by the proce-

Table 3-1

Numerical Data for Pr= 0.72, $\phi = 0^{o}$

η	f	f'	f''	θ	θ'
.000	-.000000	-.000000	.676310	1.000000	-.504760
.100	.003217	.062714	.578836	.949527	-.504641
.200	.012227	.115952	.486866	.899094	-.503849
.300	.026111	.160287	.400827	.848799	-.501806
.400	.044008	.196330	.321099	.798790	-.498052
.500	.065122	.224729	.247988	.749257	-.492239
.600	.088721	.246157	.181716	.700418	-.484143
.700	.114144	.261305	.122400	.652508	-.473660
.800	.140796	.270870	.070048	.605766	-.460798
.900	.168155	.275544	.024556	.560424	-.445673
1.000	.195764	.276003	-.014291	.516700	-.428488
1.100	.223237	.272898	-.046803	.474786	-.409519
1.200	.250246	.266841	-.073376	.434844	-.389090
1.300	.276526	.258406	-.094471	.397004	-.367557
1.400	.301865	.248113	-.110597	.361357	-.345283
1.500	.326102	.236434	-.122291	.327960	-.322626
1.600	.349119	.223785	-.130100	.296834	-.299920
1.700	.370839	.210526	-.134563	.267967	-.277468
1.800	.391215	.196966	-.136202	.241322	-.255532
1.900	.410231	.183364	-.135503	.216836	-.234332
2.000	.427893	.169929	-.132915	.194426	-.214041
2.100	.444227	.156830	-.128844	.173993	-.194791
2.200	.459275	.144197	-.123647	.155430	-.176672
2.300	.473086	.132128	-.117637	.138619	-.159741
2.400	.485721	.120688	-.111078	.123441	-.144021
2.500	.497246	.109923	-.104194	.109774	-.129510
2.600	.507729	.099854	-.097168	.097499	-.116185
2.700	.517240	.090489	-.090148	.086499	-.104006
2.800	.525850	.081820	-.083249	.076661	-.092922
2.900	.533627	.073832	-.076561	.067880	-.082873
3.000	.540638	.066499	-.070150	.060055	-.073792
3.500	.566331	.038510	-.043339	.032234	-.040536
4.000	.581001	.021663	-.025406	.017112	-.021798
4.500	.589175	.011948	-.014402	.009028	-.011582
5.000	.593654	.006500	-.007984	.004747	-.006114
5.500	.596079	.003502	-.004359	.002491	-.003215
6.000	.597382	.001874	-.002355	.001306	-.001688
6.500	.598077	.000997	-.001262	.000685	-.000885
7.000	.598446	.000529	-.000673	.000359	-.000464
7.500	.598641	.000280	-.000357	.000188	-.000243
8.000	.598744	.000147	-.000189	.000098	-.000127
9.000	.598827	.000041	-.000053	.000027	-.000035
10.000	.598850	.000011	-.000015	.000007	-.000010
11.000	.598856	.000003	-.000004	.000002	-.000003
12.000	.598857	.000001	-.000001	.000001	-.000001
14.000	.598857	.000000	-.000000	.000000	-.000000

Table 3-2

Numerical Data for Pr= 0.72, $\varphi = 10^{o}$

η	f	f'	f''	Θ	Θ'
.000	-.000000	-.000000	.668587	1.000000	-.502831
.100	.003181	.062016	.572584	.949720	-.502714
.200	.012092	.114699	.481980	.899479	-.501933
.300	.025828	.158608	.397191	.849374	-.499921
.400	.043541	.194344	.318588	.799552	-.496220
.500	.064446	.222543	.246475	.750198	-.490489
.600	.087821	.243864	.181067	.701529	-.482505
.700	.113012	.258985	.122482	.653776	-.472160
.800	.139434	.268589	.070730	.607175	-.459463
.900	.166568	.273356	.025712	.561958	-.444523
1.000	.193966	.273950	-.012778	.518339	-.427539
1.100	.221241	.271009	-.045039	.476509	-.408780
1.200	.248070	.265138	-.071456	.436631	-.388563
1.300	.274189	.256898	-.092476	.398833	-.367238
1.400	.299388	.246806	-.108596	.363208	-.345163
1.500	.323504	.235325	-.120340	.329813	-.322691
1.600	.346420	.222866	-.128242	.298672	-.300152
1.700	.368057	.209787	-.132830	.269775	-.277848
1.800	.388367	.196394	-.134614	.243085	-.256038
1.900	.407334	.182942	-.134074	.218543	-.234941
2.000	.424961	.169642	-.131649	.196067	-.214733
2.100	.441273	.156662	-.127741	.175562	-.195545
2.200	.456308	.144131	-.122702	.156921	-.177470
2.300	.470117	.132148	-.116841	.140029	-.160565
2.400	.482759	.120782	-.110421	.124768	-.144857
2.500	.494296	.110075	-.103664	.111018	-.130345
2.600	.504796	.100054	-.096751	.098660	-.117008
2.700	.514330	.090725	-.089831	.087578	-.104809
2.800	.522964	.082084	-.083020	.077662	-.093698
2.900	.530769	.074115	-.076408	.068804	-.083617
3.000	.537809	.066794	-.070060	.060906	-.074501
3.500	.563651	.038795	-.043434	.032780	-.041047
4.000	.578449	.021886	-.025544	.017448	-.022135
4.500	.586718	.012105	-.014524	.009229	-.011793
5.000	.591261	.006604	-.008075	.004865	-.006241
5.500	.593728	.003568	-.004421	.002560	-.003291
6.000	.595057	.001914	-.002395	.001346	-.001732
6.500	.595767	.001021	-.001287	.000707	-.000910
7.000	.596146	.000543	-.000688	.000371	-.000478
7.500	.596347	.000288	-.000366	.000195	-.000251
8.000	.596453	.000152	-.000194	.000102	-.000132
9.000	.596538	.000042	-.000054	.000028	-.000036
10.000	.596562	.000012	-.000015	.000008	-.000010
11.000	.596568	.000003	-.000004	.000002	-.000003
12.000	.596569	.000001	-.000001	.000001	-.000001
14.000	.596569	.000000	-.000000	.000000	-.000000

Table 3-3

Numerical Data for Pr= 0.72, $\phi = 20^{\circ}$

η	f	f'	f''	θ	θ'
.000	-.000000	-.000000	.645475	1.000000	-.496967
.100	.003073	.059926	.553842	.950306	-.496856
.200	.011688	.110940	.467296	.900650	-.496110
.300	.024981	.153569	.386226	.851123	-.494187
.400	.042141	.188380	.310978	.801867	-.490647
.500	.062416	.215970	.241838	.753059	-.485160
.600	.085115	.236960	.179012	.704907	-.477506
.700	.109610	.251988	.122617	.657634	-.467577
.800	.135336	.261699	.072669	.611467	-.455372
.900	.161794	.266734	.029084	.566632	-.440988
1.000	.188548	.267723	-.008319	.523336	-.424608
1.100	.215223	.265268	-.039811	.481768	-.406479
1.200	.241506	.259946	-.065741	.442088	-.386903
1.300	.267135	.252292	-.086519	.404425	-.366208
1.400	.291903	.242799	-.102604	.368872	-.344736
1.500	.315648	.231912	-.114480	.335492	-.322827
1.600	.338252	.220027	-.122644	.304311	-.300800
1.700	.359632	.207491	-.127591	.275326	-.278947
1.800	.379738	.194600	-.129801	.248507	-.257525
1.900	.398549	.181607	-.129726	.223799	-.236751
2.000	.416063	.168717	-.127786	.201129	-.216801
2.100	.432301	.156099	-.124363	.180407	-.197810
2.200	.447297	.143883	-.119796	.161531	-.179875
2.300	.461095	.132168	-.114383	.144394	-.163059
2.400	.473749	.121026	-.108380	.128881	-.147393
2.500	.485320	.110504	-.102005	.114877	-.132885
2.600	.495872	.100631	-.095437	.102266	-.119519
2.700	.505469	.091418	-.088823	.090936	-.107264
2.800	.514177	.082864	-.082280	.080778	-.096076
2.900	.522063	.074957	-.075899	.071687	-.085901
3.000	.529189	.067677	-.069750	.063566	-.076679
3.500	.555481	.039664	-.043702	.034498	-.042632
4.000	.570671	.022573	-.025956	.018511	-.023189
4.500	.579233	.012592	-.014896	.009869	-.012457
5.000	.583976	.006927	-.008356	.005243	-.006645
5.500	.586573	.003773	-.004614	.002780	-.003531
6.000	.587983	.002040	-.002520	.001472	-.001872
6.500	.588744	.001097	-.001366	.000779	-.000992
7.000	.589152	.000588	-.000736	.000412	-.000525
7.500	.589370	.000314	-.000395	.000218	-.000278
8.000	.589487	.000167	-.000211	.000115	-.000147
9.000	.589581	.000047	-.000060	.000032	-.000041
10.000	.589608	.000013	-.000017	.000009	-.000012
11.000	.589615	.000004	-.000005	.000003	-.000003
12.000	.589616	.000001	-.000001	.000001	-.000001
13.000	.589617	.000000	-.000000	.000000	-.000000
14.000	.589616	.000000	-.000000	.000000	-.000000

Table 3-4

Numerical Data for Pr= 0.72 , $\phi = 30^{\circ}$

η	f	f'	f''	θ	θ'
.000	-.000000	-.000000	.607128	1.000000	-.486922
.100	.002893	.056452	.522636	.951310	-.486820
.200	.011016	.104681	.442729	.902656	-.486131
.300	.023571	.145164	.367752	.854120	-.484354
.400	.039808	.178408	.298016	.805835	-.481077
.500	.059029	.204951	.233777	.757965	-.475990
.600	.080593	.225354	.175224	.710704	-.468881
.700	.103914	.240190	.122470	.664260	-.459639
.800	.128465	.250043	.075540	.618848	-.448250
.900	.153776	.255491	.034376	.574679	-.434791
1.000	.179436	.257106	-.001171	.531954	-.419416
1.100	.205088	.255438	-.031324	.490852	-.402346
1.200	.230431	.251011	-.056380	.451532	-.383843
1.300	.255215	.244320	-.076690	.414120	-.364222
1.400	.279235	.235819	-.092649	.378714	-.343780
1.500	.302332	.225921	-.104683	.345380	-.322837
1.600	.324385	.214999	-.113228	.314153	-.301695
1.700	.345309	.203378	-.118726	.285039	-.280631
1.800	.365047	.191341	-.121605	.258016	-.259894
1.900	.383571	.179130	-.122275	.233042	-.239696
2.000	.400874	.166947	-.121120	.210053	-.220215
2.100	.416967	.154955	-.118491	.188970	-.201588
2.200	.431876	.143287	-.114704	.169703	-.183921
2.300	.445638	.132044	-.110038	.152151	-.167284
2.400	.458301	.121301	-.104737	.136210	-.151718
2.500	.469917	.111111	-.099008	.121771	-.137241
2.600	.480543	.101507	-.093027	.108726	-.123847
2.700	.490239	.092509	-.086939	.096966	-.111516
2.800	.499065	.084119	-.080861	.086388	-.100212
2.900	.507083	.076333	-.074886	.076891	-.089891
3.000	.514351	.069136	-.069085	.068379	-.080500
3.500	.541401	.041151	-.044085	.037644	-.045460
4.000	.557269	.023774	-.026632	.020482	-.025097
4.500	.566347	.013458	-.015530	.011069	-.013674
5.000	.571450	.007510	-.008846	.005959	-.007395
5.500	.574284	.004148	-.004958	.003201	-.003983
6.000	.575843	.002275	-.002747	.001718	-.002140
6.500	.576696	.001240	-.001510	.000921	-.001148
7.000	.577161	.000674	-.000825	.000494	-.000616
7.500	.577412	.000365	-.000449	.000265	-.000330
8.000	.577548	.000197	-.000243	.000142	-.000177
9.000	.577661	.000057	-.000071	.000041	-.000051
10.000	.577693	.000016	-.000021	.000012	-.000015
11.000	.577702	.000005	-.000006	.000003	-.000004
13.000	.577705	.000000	-.000000	.000000	-.000000

Table 3-5

Numerical Data for Pr= 0.72 , $\phi = 40^{\circ}$

η	f	f'	f''	Θ	Θ'
.000	-.000000	-.000000	.553748	1.000000	-.472209
.100	.002643	.051604	.478954	.952781	-.472118
.200	.010078	.095923	.408084	.905594	-.471508
.300	.021598	.133363	.341426	.858512	-.469929
.400	.036536	.164358	.279240	.811652	-.467013
.500	.054270	.189368	.221747	.765164	-.462475
.600	.074226	.208870	.169111	.719218	-.456115
.700	.095877	.223356	.121437	.674005	-.447821
.800	.118746	.233325	.078761	.629720	-.437563
.900	.142408	.239275	.041050	.586556	-.425394
1.000	.166484	.241698	.008200	.544700	-.411433
1.100	.190646	.241072	-.019959	.504323	-.395859
1.200	.214612	.237855	-.043654	.465575	-.378897
1.300	.238145	.232481	-.063164	.428581	-.360805
1.400	.261049	.225351	-.078806	.393442	-.341856
1.500	.283169	.216837	-.090923	.360229	-.322329
1.600	.304382	.207273	-.099877	.328986	-.302499
1.700	.324598	.196955	-.106035	.299731	-.282620
1.800	.343757	.186147	-.109759	.272456	-.262927
1.900	.361819	.175073	-.111400	.247132	-.243625
2.000	.378769	.163925	-.111289	.223712	-.224888
2.100	.394607	.152863	-.109736	.202131	-.206858
2.200	.409349	.142017	-.107020	.182313	-.189646
2.300	.423021	.131489	-.103396	.164171	-.173335
2.400	.435660	.121361	-.099084	.147614	-.157977
2.500	.447308	.111689	-.094280	.132543	-.143602
2.600	.458014	.102515	-.089149	.118860	-.130222
2.700	.467829	.093865	-.083830	.106466	-.117826
2.800	.476805	.085752	-.078440	.095263	-.106395
2.900	.484997	.078177	-.073071	.085156	-.095895
3.000	.492458	.071134	-.067800	.076054	-.086286
3.500	.520583	.043315	-.044458	.042760	-.049865
4.000	.537456	.025585	-.027542	.023746	-.028139
4.500	.547321	.014799	-.016447	.013091	-.015656
5.000	.552987	.008434	-.009583	.007187	-.008640
5.500	.556200	.004757	-.005490	.003936	-.004746
6.000	.558005	.002662	-.003107	.002153	-.002600
6.500	.559012	.001481	-.001744	.001177	-.001422
7.000	.559571	.000821	-.000972	.000643	-.000777
7.500	.559881	.000453	-.000539	.000351	-.000425
8.000	.560051	.000250	-.000298	.000192	-.000232
9.000	.560196	.000075	-.000090	.000057	-.000069
10.000	.560240	.000022	-.000027	.000017	-.000021
11.000	.560252	.000007	-.000008	.000005	-.000006
12.000	.560256	.000002	-.000002	.000001	-.000002
14.000	.560256	.000000	-.000000	.000000	-.000000

Table 3-6

Numerical Data for Pr= 0.72 , $\phi = 45^{\circ}$

η	f	f'	f''	θ	θ'
.000	-.000000	-.000000	.521470	1.000000	-.462848
.100	.002491	.048665	.452397	.953717	-.462764
.200	.009509	.090599	.386869	.907463	-.462200
.300	.020398	.126167	.325142	.861307	-.460738
.400	.034543	.155762	.267448	.815357	-.458036
.500	.051366	.179799	.213986	.769750	-.453825
.600	.070332	.198706	.164906	.724649	-.447912
.700	.090951	.212930	.120307	.680228	-.440186
.800	.112777	.222919	.080231	.636673	-.430611
.900	.135408	.229127	.044655	.594166	-.419222
1.000	.158491	.231998	.013498	.552886	-.406121
1.100	.181711	.231969	-.013379	.512994	-.391464
1.200	.204802	.229459	-.036169	.474638	-.375452
1.300	.227534	.224864	-.055108	.437941	-.358314
1.400	.249718	.218557	-.070471	.403004	-.340302
1.500	.271199	.210880	-.082556	.369901	-.321674
1.600	.291858	.202144	-.091682	.338681	-.302684
1.700	.311602	.192631	-.098173	.309368	-.283575
1.800	.330367	.182587	-.102352	.281962	-.264569
1.900	.348109	.172227	-.104534	.256444	-.245865
2.000	.364808	.161737	-.105022	.232773	-.227636
2.100	.380457	.151270	-.104096	.210896	-.210023
2.200	.395067	.140956	-.102016	.190744	-.193141
2.300	.408657	.130897	-.099018	.172240	-.177076
2.400	.421258	.121176	-.095309	.155299	-.161888
2.500	.432906	.111853	-.091075	.139832	-.147617
2.600	.443643	.102973	-.086473	.125745	-.134279
2.700	.453516	.094566	-.081640	.112945	-.121875
2.800	.462573	.086649	-.076688	.101339	-.110391
2.900	.470862	.079229	-.071712	.090837	-.099802
3.000	.478435	.072305	-.066786	.081350	-.090074
3.500	.507216	.044674	-.044569	.046358	-.052830
4.000	.524731	.026764	-.028063	.026082	-.030235
4.500	.535116	.015696	-.017016	.014564	-.017050
5.000	.541163	.009067	-.010060	.008096	-.009532
5.500	.544637	.005181	-.005844	.004489	-.005302
6.000	.546615	.002938	-.003353	.002485	-.002941
6.500	.547733	.001656	-.001907	.001375	-.001629
7.000	.548362	.000929	-.001077	.000760	-.000901
7.500	.548714	.000519	-.000605	.000420	-.000498
8.000	.548911	.000290	-.000339	.000232	-.000275
9.000	.549081	.000089	-.000105	.000071	-.000084
10.000	.549133	.000027	-.000033	.000022	-.000026
11.000	.549149	.000008	-.000010	.000007	-.000008
12.000	.549153	.000002	-.000003	.000002	-.000002
14.000	.549154	.000000	-.000000	.000000	-.000000

Table 3-7

Numerical Data for Pr= 0.72 , $\phi = 60^\circ$

η	f	f'	f''	θ	θ'
.000	-.000000	-.000000	.402084	1.000000	-.424419
.100	.001928	.037744	.353146	.957560	-.424360
.200	.007389	.070706	.306492	.915139	-.423958
.300	.015918	.099124	.262276	.872789	-.422914
.400	.027071	.123248	.220639	.830587	-.420972
.500	.040433	.143342	.181706	.788632	-.417929
.600	.055614	.159683	.145578	.747043	-.413627
.700	.072254	.172554	.112329	.705952	-.407962
.800	.090019	.182247	.082001	.665498	-.400879
.900	.108607	.189053	.054605	.625823	-.392374
1.000	.127743	.193265	.030118	.587069	-.382484
1.100	.147183	.195172	.008483	.549370	-.371292
1.200	.166710	.195054	-.010386	.512850	-.358911
1.300	.186135	.193183	-.026604	.477622	-.345486
1.400	.205297	.189817	-.040314	.443782	-.331178
1.500	.224057	.185199	-.051674	.411410	-.316164
1.600	.242302	.179555	-.060860	.380567	-.300626
1.700	.259941	.173093	-.068058	.351296	-.284746
1.800	.276900	.166003	-.073462	.323623	-.268698
1.900	.293126	.158454	-.077263	.297556	-.252647
2.000	.308581	.150598	-.079653	.273088	-.236743
2.100	.323240	.142565	-.080816	.250198	-.221120
2.200	.337091	.134469	-.080928	.228851	-.205893
2.300	.350135	.126409	-.080155	.209003	-.191159
2.400	.362377	.118463	-.078648	.190600	-.176995
2.500	.373833	.110699	-.076546	.173583	-.163462
2.600	.384525	.103169	-.073975	.157885	-.150603
2.700	.394476	.095916	-.071045	.143439	-.138447
2.800	.403718	.088969	-.067854	.130172	-.127009
2.900	.412281	.082351	-.064486	.118013	-.116292
3.000	.420199	.076075	-.061013	.106889	-.106290
3.500	.451339	.049920	-.043861	.064405	-.066308
4.000	.471451	.031718	-.029576	.038257	-.040251
4.500	.484087	.019698	-.019124	.022521	-.024013
5.000	.491872	.012034	-.012018	.013184	-.014171
5.500	.496601	.007265	-.007404	.007692	-.008308
6.000	.499444	.004347	-.004497	.004478	-.004851
6.500	.501139	.002584	-.002703	.002604	-.002826
7.000	.502144	.001528	-.001612	.001513	-.001644
7.500	.502737	.000900	-.000956	.000879	-.000955
8.000	.503086	.000528	-.000564	.000510	-.000555
9.000	.503409	.000181	-.000195	.000172	-.000187
10.000	.503519	.000061	-.000067	.000058	-.000063
11.000	.503556	.000020	-.000023	.000019	-.000021
12.000	.503568	.000007	-.000008	.000006	-.000007
14.000	.503572	.000000	-.000001	.000000	-.000001

Table 3-8

Numerical Data for Pr= 0.72 , $\varphi = 70^{\circ}$

η	f	f'	f''	Θ	Θ'
.000	-.000000	.000000	.302416	1.000000	-.385968
.100	.001456	.028553	.268874	.961404	-.385927
.200	.005601	.053822	.236745	.922822	-.385651
.300	.012115	.075953	.206115	.884289	-.384929
.400	.020692	.095098	.177066	.845857	-.383580
.500	.031041	.111421	.149673	.807599	-.381454
.600	.042888	.125090	.123998	.769597	-.378429
.700	.055976	.136280	.100092	.731946	-.374417
.800	.070067	.145169	.077988	.694748	-.369359
.900	.084940	.151939	.057705	.658110	-.363230
1.000	.100390	.156771	.039246	.622138	-.356033
1.100	.116235	.159848	.022593	.586937	-.347800
1.200	.132308	.161349	.007715	.552610	-.338586
1.300	.148458	.161449	-.005437	.519250	-.328471
1.400	.164556	.160317	-.016926	.486943	-.317548
1.500	.180486	.158117	-.026830	.455764	-.305928
1.600	.196149	.155001	-.035235	.425776	-.293729
1.700	.211461	.151116	-.042237	.397033	-.281073
1.800	.226351	.146597	-.047940	.369573	-.268087
1.900	.240763	.141568	-.052451	.343422	-.254892
2.000	.254652	.136143	-.055878	.318597	-.241607
2.100	.267982	.130425	-.058332	.295100	-.228342
2.200	.280730	.124506	-.059919	.272925	-.215198
2.300	.292879	.118467	-.060743	.252053	-.202266
2.400	.304422	.112379	-.060906	.232461	-.189626
2.500	.315355	.106305	-.060500	.214116	-.177344
2.600	.325685	.100295	-.059614	.196979	-.165477
2.700	.335418	.094395	-.058328	.181005	-.154071
2.800	.344569	.088641	-.056716	.166148	-.143158
2.900	.353152	.083061	-.054846	.152356	-.132764
3.000	.361187	.077678	-.052776	.139577	-.122904
3.500	.393904	.054172	-.041053	.088989	-.081659
4.000	.416346	.036539	-.029754	.055866	-.052677
4.500	.431304	.024057	-.020579	.034709	-.033310
5.000	.441066	.015560	-.013783	.021418	-.020788
5.500	.447340	.009933	-.009024	.013158	-.012863
6.000	.451326	.006278	-.005812	.008061	-.007916
6.500	.453837	.003938	-.003698	.004930	-.004855
7.000	.455407	.002456	-.002332	.003011	-.002971
7.500	.456383	.001524	-.001460	.001838	-.001816
8.000	.456989	.000942	-.000910	.001121	-.001109
9.000	.457592	.000357	-.000349	.000416	-.000413
10.000	.457819	.000133	-.000132	.000154	-.000154
11.000	.457903	.000049	-.000050	.000057	-.000057
12.000	.457933	.000017	-.000018	.000020	-.000021
14.000	.457946	.000001	-.000002	.000002	-.000003

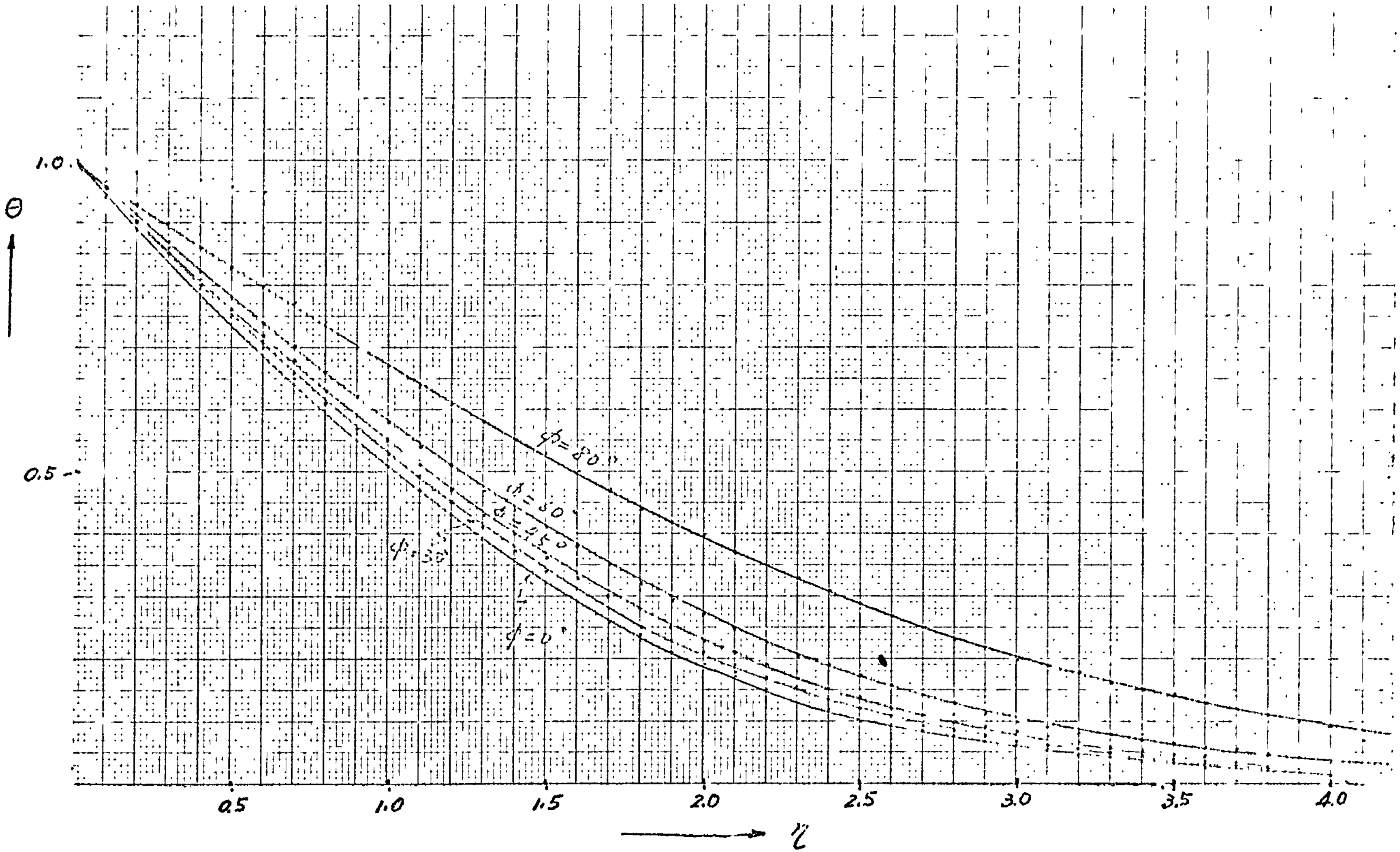

Figure 3-1 Dimensionless temperature distributions for various angles of inclination (Pr=0.72)

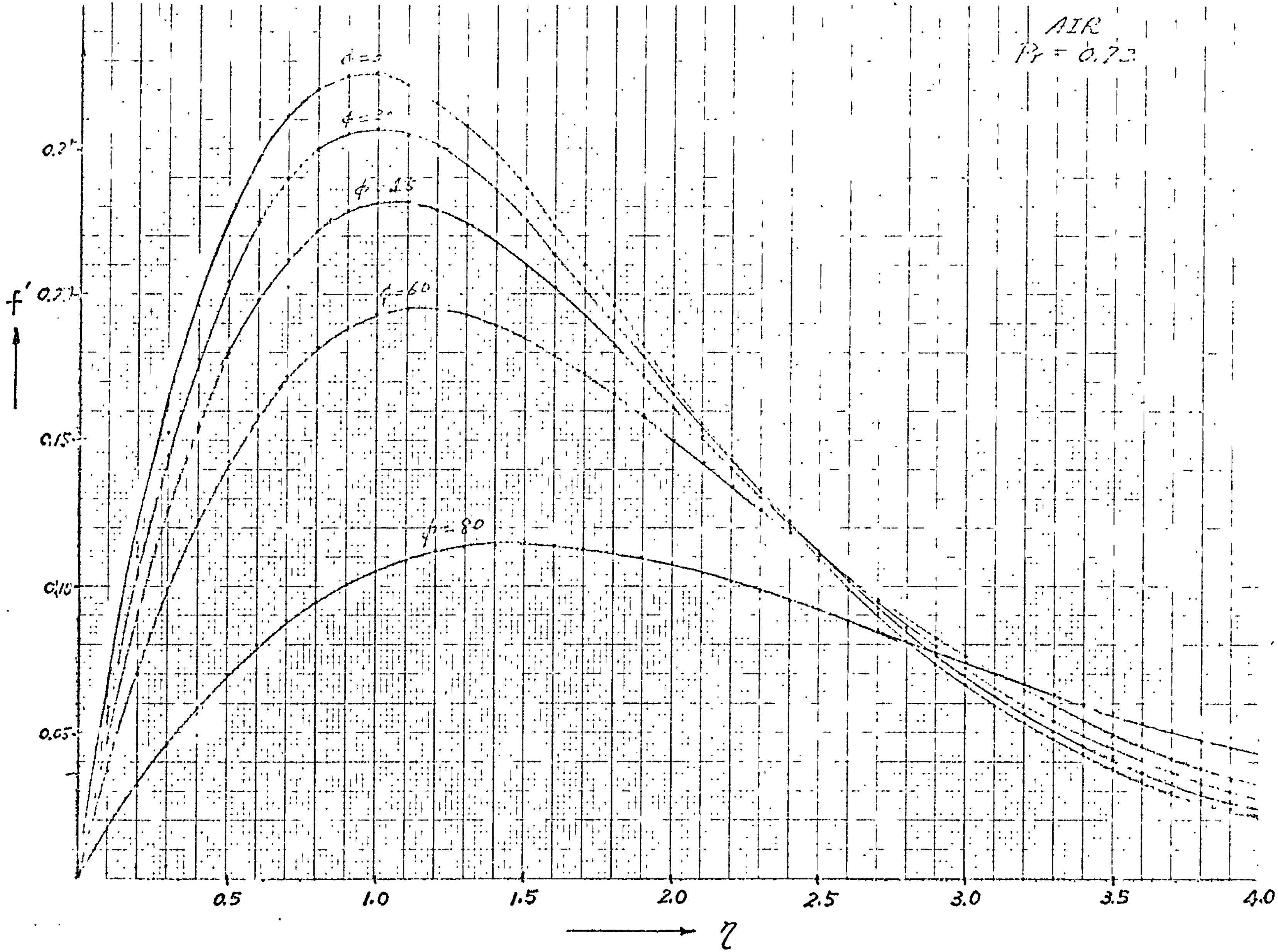

Figure 3-2 Dimensionless velocity distributions for various angles of inclination (Pr=0.72)

dures described. $\eta = 15$ was used to represent infinity and a step size 0.05 was used in computation. The numerical data f, f', f", Θ and Θ' are shown in Table 3-1 to Table 3-8. The profiles of dimensionless temperature distribution, Θ, and dimensionless velocity distribution, f', versus variable η are ploted in Figure 3-1 and Figure 3-2 respectively for the different angles of inclination.

The velocity components can now be calculted by following equations :

$$u = 4\nu x^{\frac{1}{2}} c^2 f'(\eta)$$
$$v = \nu c x^{-\frac{1}{4}} [\eta f'(\eta) - 3 f(\eta)] \qquad (3\text{-}14)$$

for certain specified coordinates (x, y).

The method used for the numerical integration is Runge-Kutta fourth order method.

The local Nusselt number is defined as :

$$Nu_x \equiv \frac{hx}{K} = \frac{-x}{(t_\omega - t_\infty)} \left.\frac{\partial t}{\partial y}\right|_{y=0} \qquad (3\text{-}15)$$

To express $\frac{\partial t}{\partial y}$ in term of known function, $\theta'(\eta)$, equation (2-18) is used. Thus:

$$\frac{\partial t}{\partial y} = \theta(\eta)\, c\, x^{-\frac{1}{4}} (t_\omega - t_\infty) \qquad (3\text{-}16)$$

Substitution of this expression into equation (3-15) yields the local Nusselt number as :

$$Nu_x = -c\, x^{\frac{3}{4}} \theta'(0) \qquad (3\text{-}17)$$

A dimensionless heat transfer parameter may then be defined as :

$$\frac{Nu_x}{CX^{3/4}} = -\theta'(0) \qquad (3\text{-}18)$$

The computations were also made for a wide ranges of Prandtl numbers. The nondimensional heat transfer parameters ,$-\theta'(0)$, were obtained for Prandtl numbers equal to 0.01, 0.72, 0.733, 1.0, 10, 100, and 200 for the angles of inclination ranging from 0° to 40°. A step size of 0.05 was also used for all these calculations.

It is found that $\eta = 15$ can represent infinity satisfactory for the low Prandtl number cases, and $\eta = 10$ is very satisfactory for the high Prandtl number cases. For very low Prandtl cases, such as Pr = 0.01, $\eta = 20$ shall be used.

The numerical results of these nondimensional heat transfer parameters are listed in Table 3-9.

According to equation (3-17), the local Nusselt numbers could also be computed as a function of local Grashof number, Gr, defined as :

$$Gr \equiv \beta g (t_w - t_\infty) X^3 / \nu^2$$

Thus, equation (3-17) becomes:

$$Nu_x = -\theta'(0) \left(\frac{Gr}{4}\right)^{\frac{1}{4}} \qquad (3\text{-}19)$$

For $\phi = 20^{\circ}$ and $\phi = 40^{\circ}$, the Nusselt numbers are ploted versus local Grashof number for some fixed Prandtl numbers and are shown in Figure 3-3 and Figure 3-4 respectively

Table 3-9

Nondimensional Heat Transfer Parameter of

Free Convection from an Inclined Isothermal Flat Plate

Pr\ ϕ	0°	10°	20°	30°	40°
0.01	0.084149	0.083888	0.083079	0.081753	0.079814
0.72	0.504760	0.502831	0.496967	0.486922	0.472209
0.733	0.508035	0.505985	0.500087	0.489983	0.475183
1.0	0.567307	0.565138	0.558548	0.547257	0.530719
10	1.170247	1.165740	1.152730	1.128815	1.094665
100	2.196541	2.187727	2.160292	2.118180	2.053873
200	2.633341	2.616709	2.592409	2.539658	2.462414

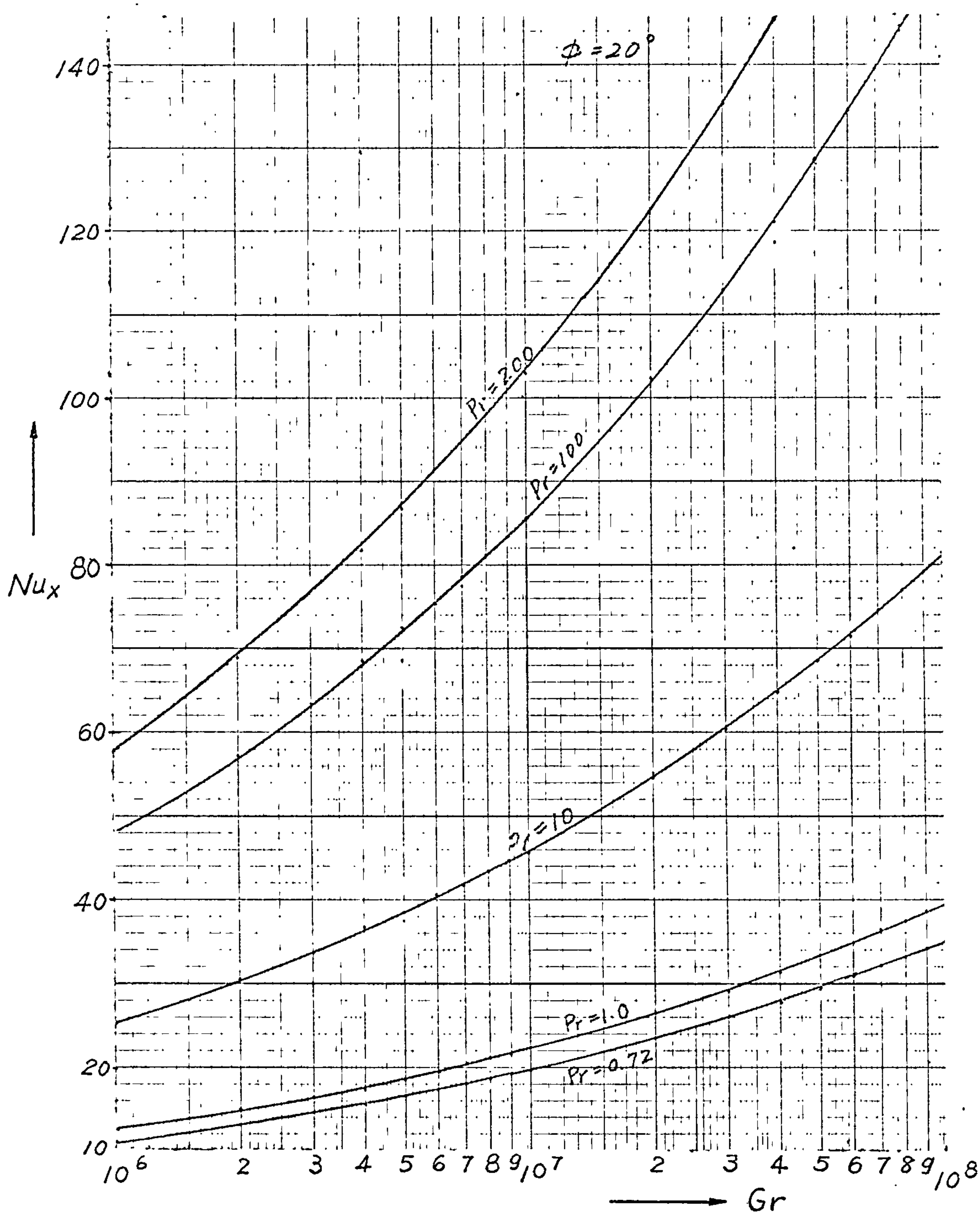

Figure 3-3

Local Nusselt Number Versus Grashof Number (Free Convection from a Heated Isothermal Flat Plate Inclined 20° from Vertical)

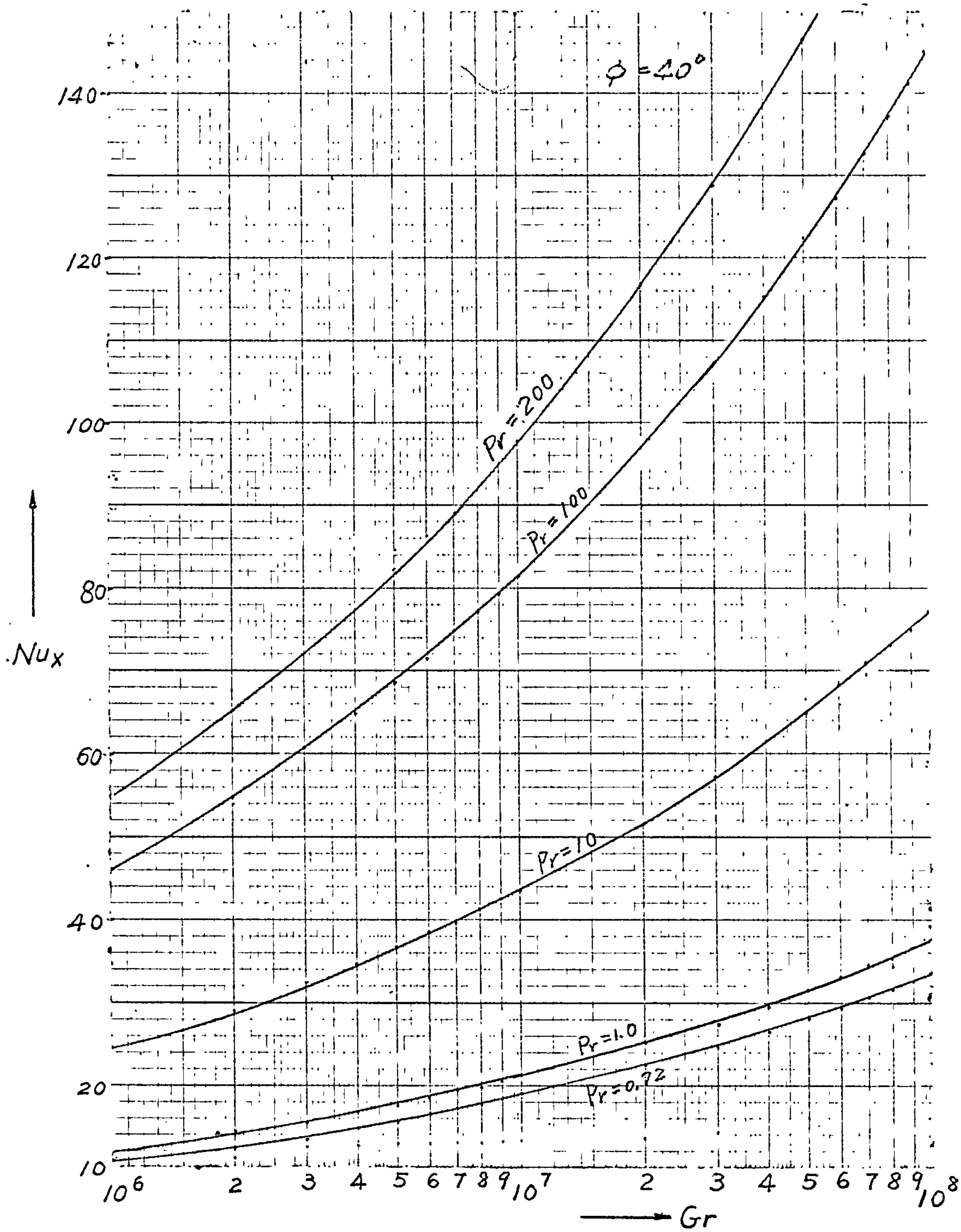

Figure 3-4

Local Nusselt Number Versus Grashof Number (Free Convection from a Heated Isothermal Flat Plate Inclined 40° from vertical)

CHAPTER IV

CONCLUSIONS

The differential equations describing the free convection heat transfer about an inclined plate have been solved by a numerical integration method. The partial differential equations were transformed into a coupled set of ordinary nonlinear differential equations via a similarity transform. These nonlinear differential equations are subject to boundary conditions at the origin and at infinity, A suitable approximation of infinity was made and the solution was the obtained via a quasilinearization procedure. The numerical solutions of these equations are vsry sensitive for large Prandtl numbers.

The following conclusions may be drawn from a careful study of the numerical results :

(1) Influence of angle of inclination ϕ on the flow and temperature gradient within the boundary layers is seen from the profiles in Figure 3-1 and Figure 3-2. For a fixed value of x, the thickness of both hydrodynamic and thermal boundary layers increases as ϕ increases. However, the maximum velocity within the boundary layer decreases as ϕ increases. These conclusions were expected from the physics of the problem because as the angle ϕ increases, the component of the bouyant force in the y-direction increases.

(2) For the cases of small and medium angles of inclination where the fluid is air, the agreement of the calculated Nusselt number with Rich's experimental results is good. A comparison of the Nusselt number versus the local Grashof-Prandtl number is shown in Figure 4-1 for the case of $\phi = 20^{\circ}$. It was expected that the numerical results are in error for large values of ϕ because the momentum in the y-direction was neglected.

(3) Boundary layer assumptions, implying that the distance along the plate is much larger than the boundary layer thickness, are valid for the cases of small ϕ.

(4) When the Prandtl number is less than 100, the empirical Nusselt number for the vertical plate (18) can be modified according to Rich's proposal as :

$$Nu_x = \left[\frac{0.676 Pr^{\frac{1}{2}}}{(0.861 + Pr)^{\frac{1}{4}}} \left(\frac{Gr}{4} \right)^{\frac{1}{4}} \right] (\cos\phi)^{\frac{1}{4}} \tag{4-1}$$

which has excellent correlation with numerical results shown in preceding chapter. The local Nusselt number can be approximated by this equation within 2% for Grashof number from 10^4 to 10^8. When the Prandtl number is larger, the approximation (4-1) is poor. For the cases of large Prandtl numbers, the modified Eckert equation (8)

$$Nu_x = \left[\frac{0.508 \, Pr^{\frac{1}{2}}}{(0.952 + Pr)^{\frac{1}{4}}} \, Gr^{\frac{1}{4}} \right] (\cos\phi)^{\frac{1}{4}} \tag{4-2}$$

has better correlation.

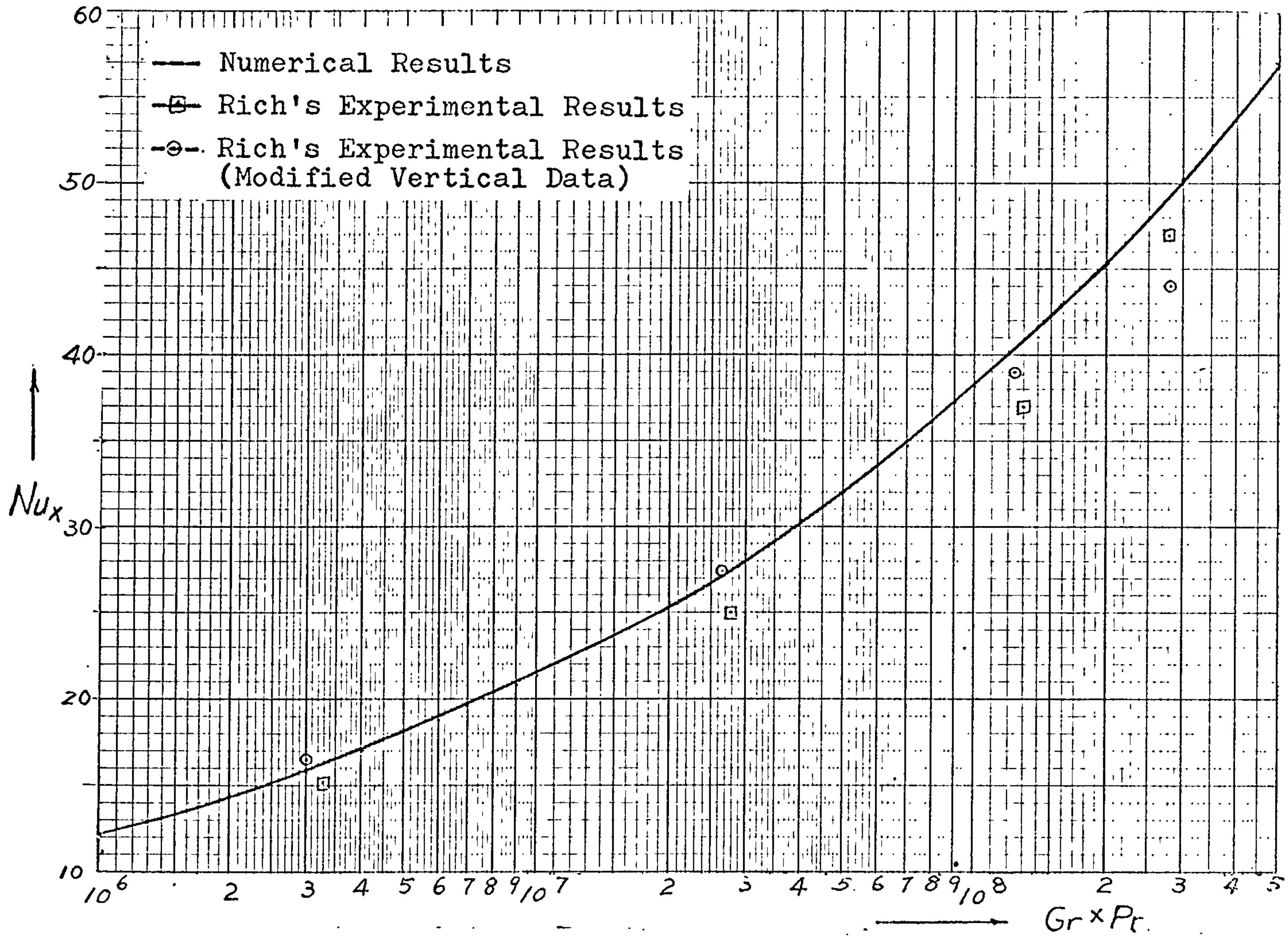

Figure 4-1

Local Nusselt Number Versus Local Grashof-Prandtl Number (inclined 20 degrees from vertical)

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