IDEALS IN NEAR-RINGS

A Thesis

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Frank J. Hall

August, 1967

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ABSTRACT

Near-rings form a class which contains rings. In this thesis, a definition for a near-ring ideal is given. Generalizations from ideals in rings are then obtained.

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CHAPTER I

INTRODUCTION

Definition of near-ring ideal

Suppose that (G,+) is an abelian group and that End(G) is the set of endomorphisms on G. We define on End(G) two binary operations (denoted by + and ·) in the following manner:

$$g(a+b) = ga + gb$$

$$g(a \cdot b) = (ga)b$$

where g belongs to G and a,b belong to End(G). One can easily verify that the system (End(G),+,·) is an example of a ring.

Now, suppose that G is not abelian. Then, not all of the ring properties hold for (End(G),+,·). For example, commutativity of addition fails. Moreover, closure with respect to addition is not guaranteed. However, if we consider the set T(G) of transformations on G (under the same binary operations), all ring properties except commutativity of addition and the right distributive law are satisfied. (T(G),+,·) is an example of a (left) near-ring. More precisely, a (left) near-ring is a system (N,+,·) such that

(i) (N,+) is a group,

(ii) (N,·) is a semigroup,

(iii) multiplication is left distributive with respect to addition.

Clearly, a ring is a near-ring.

If R is a ring, the ideals in R coincide with the kernels of homomorphisms of R. Do there exist definitions for ideals and near-ring homomorphisms such that the ideals in a near-ring N are the kernels of near-ring homomorphisms of N?

A mapping λ from a near-ring N into a near-ring N' will be called a near-ring homomorphism if

$$(n_1+n_2)\lambda = n_1\lambda + n_2\lambda \text{ and } (n_1n_2)\lambda = (n_1\lambda)(n_2\lambda),$$

where n_1, n_2 belong to N. Suppose that J is any additive normal subgroup of N. Let π denote the natural group homomorphism from N onto N/J. Define multiplication on N/J in the following way:

$$(n_1+J)(n_2+J) = n_1n_2 + J.$$

Under which conditions will this multiplication be well-defined? Suppose that $n_1 \equiv n_1'$ (mod J) and $n_2 \equiv n_2'$ (mod J). Now, $n_1 = n_1' + j_1$ for some j_1 in J. Since J is normal in (N,+), $n_2' + J = J + n_2'$. So, $n_2 = j_2 + n_2'$ for some j_2 in J.

Hence,

$$n_1n_2 = (n_1'+j_1)(j_2+n_2')$$
$$= (n_1'+j_1)j_2 + (n_1'+j_1)n_2'$$
$$= (n_1'+j_1)j_2 + (n_1'+j_1)n_2' - n_1'n_2' + n_1'n_2'.$$

Assume that NJ is a subset of J and that $(n+j)n' - nn'$ is in J for n, n' in N and j in J. Then, $(n_1'+j_1)j_2$ and $(n_1'+j_1)n_2' - n_1'n_2'$ belong to J. Thus,

$$n_1n_2 = j_3 + n_1'n_2' \text{ for some } j_3 \text{ in J.}$$

Hence, $n_1n_2 \equiv n_1'n_2'$ (mod J) and the multiplication is well-defined. Consequently, N/J is a near-ring and π is a near-ring homomorphism. J is the kernel of π.

Let K be the kernel of a homomorphism λ of a near-ring N. Then, K is an additive normal subgroup of N. Since $n0 = n(0+0) = n0 + n0$, $n0 = 0$ for any n in N. So NK is a subset of K. Also,

$$[(n_1+k)n_2 - n_1n_2]\lambda = [(n_1+k)n_2]\lambda - (n_1n_2)\lambda$$
$$= [(n_1+k)\lambda](n_2\lambda) - (n_1n_2)\lambda$$
$$= (n_1\lambda+k\lambda)(n_2\lambda) - (n_1n_2)\lambda$$
$$= (n_1\lambda)(n_2\lambda) - (n_1n_2)\lambda = 0.$$

Thus, $(n_1+k)n_2 - n_1n_2$ is in K for n_1n_2 in N, k in K.

We define an ideal J in N as an additive normal subgroup such that NJ is a subset of J and $(n_1+j)n_2 - n_1n_2$ belongs to J for n_1, n_2 in N and j in J (see [1]). Therefore, ideals in N coincide with the kernels of near-ring homomorphisms of N. Note that if N is a ring, an ideal in N (considered as a near-ring) is an ideal in N considered as a ring.

An example of an ideal

We give an example of an ideal which will be used for later reference. Consider the non-abelian group of order 6.

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	b	0	e	c	d
b	b	0	a	d	e	c
c	c	d	e	0	a	b
d	d	e	c	b	0	a
e	e	c	d	a	b	0

Let M = {0,a,b} and T = {c,d,e}. If m belongs to M and x belongs to G, define mx = 0. If y belongs to T and x belongs to G, define yx = x. Under this multiplication, G is a near-ring (see [3]). Notice that (c+e)a = ba = 0 and ca + ea = a + a = b; the right distributive law does not hold.

Now, M is a normal subgroup of (G,+) and MG is a subset of M. Also, $(x_1+m)x_2 - x_1x_2 = 0$ for m in M and x_1, x_2 in G. If x_1 belongs to M, then $(x_1+m)x_2 - x_1x_2 = 0 - 0 = 0$. If x_1 belongs to T, $x_1 + m$ belongs to T for m in M. In this case, $(x_1+m)x_2 - x_1x_2 = 0$. Hence, M is an ideal in G. As we shall see in Chapter II, the ideal M is also principal and prime.

CHAPTER II

GENERALIZATIONS FROM RING THEORY

Sum of ideals

Suppose that J and L are ideals of a near-ring N. Let $J + L = \{j+l \mid j \text{ is in } J, l \text{ is in } L\}$. Is J + L an ideal? Suppose $j_1 + l_1$ and $j_2 + l_2$ are in J + L. Then,

$$(j_1+l_1) - (j_2+l_2) = j_1 + l_1 - l_2 - j_2$$

$$= j_1 + l_3 - j_2 \text{ for some } l_3 \text{ in } L$$

$$= j_1 + j_3 + l_3 \text{ for some } j_3 \text{ in } J$$

$$\text{since } l_3 + J = J + l_3$$

$$= j_4 + l_3 \text{ for some } j_4 \text{ in } J.$$

So, J + L is a subgroup of (N,+). Since,

$$n + j + l - n = n + j - n + n + l - n \text{ is in } J + L$$

for n in N, j in J, and l in L, J + L is a normal subgroup of (N,+). Now, $n(j+l) = nj + nl$, which is in J + L. Hence, N(J+L) is a subset of J + L. Consider

$$[(n_1+j) + l]n_2 - (n_1+j)n_2 = l_1 \text{ for some } l_1 \text{ in } L.$$

Now,

$$[(n_1+j) + l]n_2 = l_1 + (n_1+j)n_2 - n_1n_2 + n_1n_2.$$

Hence,

$$[(n_1+j) + 1]n_2 - n_1n_2 = 1_1 + j_1 \text{ for some } j_1 \text{ in } J.$$

But, $1_1 + j_1$ is in J + L since J and L are normal in (N,+). So, J + L is an ideal. By induction, a finite sum of ideals is an ideal.

A decomposition theorem

Suppose that (N,+,·) is a near-ring. A sub-near-ring of (N,+,·) is a system (Q,+,·) such that (Q,+) is a subgroup of (N,+) and (Q,·) is a sub-semi-group of (N,·). The near-ring (N,+,·) is said to be the sum of the sub-near-rings (S,+,·) and (T,+,·) iff every element of N can be expressed as a unique sum of elements s + t, s in S, t in T.

Let $N_c = \{a \text{ in } N | 0a=0\}$ and $N_z = \{z \text{ in } N | az=z, a \text{ in } N\}$. Then N_c and N_z are sub-near-rings and N is the sum of N_c and N_z. Note that the intersection of N_c and N_z is {0} (see [1]). Suppose that J is an ideal in N.
Let $J_c = \{n_c \text{ in } N_c | n_c + n_z \text{ is in } J \text{ for some } n_z \text{ in } N\}$ and $J_z = \{n_z \text{ in } N_z | n_c + n_z \text{ is in } J \text{ for some } n_c \text{ in } N_c\}$. We will show that $J = J_c + J_z$ and that J_c and J_z are ideals in N_c and N_z respectively.

Since J is an ideal in N, J is the kernel of some homomorphism λ from N onto a near-ring N'. Let λ restricted to N_c be denoted by λ_1 and λ restricted to N_z be denoted by

λ_2. λ_1 and λ_2 are homomorphisms into N' of N_C and N_Z, respectively. Since, for n_C in N_C,

$$0 = 0\lambda = (0n_C)\lambda = (0\lambda)(n_C\lambda) = 0(n_C\lambda),$$

$N_C\lambda$ is a subset of N_C'. If q belongs to N', there exists n in N such that $n\lambda = q$. So, $q(n_Z\lambda) = (n\lambda)(n_Z\lambda) = (nn_Z)\lambda = n_Z\lambda$ if n_Z is in N_Z. Hence, $N_Z\lambda$ is a subset of N_Z'.

Let J_1 be the kernel of λ_1 and J_2 be the kernel of λ_2. If n_C belongs to J_1,

$$(n_C+0)\lambda = n_C\lambda_1 + 0\lambda_2 = 0 + 0 = 0.$$

So, n_C is in J_C. If n_Z belongs to J_2,

$$(0+n_Z)\lambda = 0\lambda_1 + n_Z\lambda_2 = 0 + 0 = 0.$$

So, n_Z is in J_Z. Now, if n_C belongs to J_C, $n_C + n_Z$ belongs to J for some n_Z in N_Z. Hence, $(n_C+n_Z)\lambda = 0$. So, $n_C\lambda_1 + n_Z\lambda_2 = 0$. Now, $n_C\lambda_1$ belongs to N_C' and $n_Z\lambda_2$ belongs to N_Z'. Since every element of N' has a unique expression as a sum of elements from N_C' and N_Z', $n_C\lambda_1 = n_Z\lambda_2 = 0$. Thus, n_C belongs to J_1 and J_C is a subset of J_1. Similarly, J_Z is a subset of J_2. Consequently, $J_C = J_1$, $J_Z = J_2$ and J_C and J_Z are ideals in N_C and N_Z, respectively.

If j belongs to J, $j = n_C + n_Z$ where n_C belongs to N_C and n_Z belongs to N_Z. So, J is a subset of $J_C + J_Z$. If $n_C + n_Z$ belongs to $J_C + J_Z$, $(n_C+n_Z)\lambda = n_C\lambda_1 + n_Z\lambda_2 = 0 + 0 = 0.$

So, $J_c + J_z$ is a subset of J. Therefore, $J = J_c + J_z$.

J_c and J_z are not necessarily ideals in N. Suppose that J_c is an ideal in N_c and J_z is an ideal in N_z. Is $J_c + J_z$ an ideal in N? We give an example to show that this is not always the case.

If N and N' are near-rings, the cartesian product N × N' is a near-ring under component-wise addition and multiplication. Let each of (N,+) and (N',+) be the cyclic group of order 6.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Let the following be, respectively, the multiplication tables for (N,·) and (N',·):

·	0	1	2	3	4	5
0	0	3	0	3	0	3
1	0	3	0	3	0	3
2	0	1	2	3	4	5
3	0	3	0	3	0	3
4	0	1	2	3	4	5
5	0	1	2	3	4	5,

·	0	1	2	3	4	5
0	0	4	2	0	4	2
1	0	4	2	0	4	2
2	0	4	2	0	4	2
3	0	4	2	0	4	2
4	0	4	2	0	4	2
5	0	4	2	0	4	2.

The systems $(N,+,\cdot)$ and $(N',+,\cdot)$ are near-rings (see [2]). So $N \times N'$ is a near-ring.

Now,

$$(N\times N')_c = \{(0,0),(2,0),(4,0),(0,3),(2,3),(4,3)\}$$

and $$(N\times N')_z = \{(0,0),(3,0),(0,2),(3,2),(0,4),(3,4)\}.$$

The sets $J_c = \{(0,0),(0,3)\}$ and $J_z = \{(0,0),(3,0)\}$ are proper ideals in $(N\times N')_c$ and $(N\times N')_z$, respectively.

Since

$$[(1,0) + (3,3)](2,0) - (1,0)(2,0)$$

$$= (4,3)(2,0) - (1,0)(2,0)$$

$$= (2,0) - (0,0) = (2,0),$$

$J_c + J_z$ is not an ideal in $N \times N'$.

Prime ideals

A. The intersection of an arbitrary number of ideals of a near-ring N is clearly an ideal. By the ideal generated by a subset S of N, we shall mean the intersection of all the ideals of N which contain S. This ideal is denoted by (S). The ideal generated by an arbitrary element of N is called a principal ideal. If A and B are two ideals in N, the product AB is defined to be the ideal generated by the set of all products ab, a in A, b in B. An ideal P in N is a prime ideal iff A and B are ideals in N such that if A is not a subset of P, B is not a subset of P, then AB is not a subset of P. This definition generalizes the definition of a prime ideal for a ring (see [4]). Notice that the ideal M of Section 2 in Chapter I is prime. M is the only non-zero proper ideal of G. So, the only ideal which is not a subset of M is G. But, GG is not a subset of M. Hence, M is a prime ideal. Moreover, M is a principal ideal; M is generated by a or by b.

The following proposition was stated without proof in [5]. An ideal P in N is prime iff any one of the following conditions is satisfied:

(i) If a, b do not belong to P, then (a)(b) is not a subset of P.

(ii) If a, b do not belong to P, there exist an element a_1 in (a) and an element b_1 in (b) such that

a_1b_1 is not in P.

(iii) If A and B are ideals in N which properly contain P, then AB is not a subset of P.

We give the proof here. Suppose that P is a prime ideal in N and that a, b do not belong to P. So, (a) is not a subset of P and (b) is not a subset of P. Therefore, (a)(b) is not a subset of P. Thus, if P is a prime ideal, (i) holds.

Suppose that (i) is satisfied and a, b do not belong to P. Then (a)(b) is not a subset of P. If for each a_1 in (a), b_1 in (b), a_1b_1 is in P, then P contains (a)(b). So there exist an element a_1 in (a) and an element b_1 in (b) such that a_1b_1 does not belong to P. Thus, (ii) holds.

Suppose that (ii) holds and A and B are ideals in N which properly contain P. Then there exist an element a in A and an element b in B such that a, b do not belong to P. So there exists a_1 in (a) and b_1 in (b) such that a_1b_1 is not in P. Now, (a) is a subset of A and (b) is a subset of B. Now, a_1b_1 is in (a)(b). But, (a)(b) is a subset of AB. Hence, AB is not a subset of P. Thus, (iii) is satisfied.

Finally, assume that (iii) holds. Suppose that P is not a prime ideal. Then, there exist ideals A, B such that A is not a subset of P, B is not a subset of P and

AB is a subset of P. So, there exist a in A, b in B such that a, b are not in P. Now, (a) is a subset of A and (b) is a subset of B. Hence, (a)(b) is a subset of P. Now, (a) + P and (b) + P are two ideals which properly contain P. By (iii), [(a) + P][(b) + P] is not a subset of P. So, there exist $x_1 + x_2$ in (a) + P, $y_1 + y_2$ in (b) + P such that $(x_1+x_2)(y_1+y_2)$ is not in P. Now,

$$(x_1+x_2)(y_1+y_2) = (x_1+x_2)y_1 + (x_1+x_2)y_2.$$

Since y_2 belongs to P and P is an ideal, $(x_1+x_2)y_2$ belongs to P. Since x_2 belongs to P and P is an ideal, $(x_1+x_2)y_1 - x_1y_1$ is in P. Also, x_1y_1 is in (a)(b) which is a subset of P. Therefore, $(x_1+x_2)y_1$ belongs to P. So, $(x_1+x_2)y_1 + (x_1+x_2)y_2$ belongs to P. This is a contradiction. Hence, P is a prime ideal.

B. If R is a ring with commutative multiplication, an ideal P in R is prime iff ab belongs to P, then a belongs to P or b belongs to P. We will show the same result holds for a near-ring N with commutative multiplication.

Suppose that J is an additive normal subgroup of (N,+) such that NJ is a subset of J. Now,

$$\begin{aligned}(n_1+j)n_2 - n_1n_2 &= n_2(n_1+j) - n_1n_2 \\ &= n_2n_1 + n_2j - n_1n_2 \\ &= n_1n_2 + n_2j - n_1n_2\end{aligned}$$

belongs to J for n_1, n_2 in N, j in J, since J is an additive normal subgroup of (N,+). Hence, J is an ideal in N.

Suppose that A and B are ideals in N. Let

$$T = \{\sum_{\text{finite}} (d_i+a_ib_i-d_i) \mid a_i \text{ is in } A,\ b_i \text{ in } B,\ d_i \text{ is in } N\}.$$

We will show that AB = T. Now, T is clearly a subgroup of (N,+). If $\sum_{\text{finite}} (d_i+a_ib_i-d_i)$ is in T,

$$n + \sum_{\text{finite}} (d_i+a_ib_i-d_i) - n$$

$$= n + \left[\sum_{\text{finite}} (d_i+a_ib_i-d_i-n+n)\right] - n$$

$$= \sum_{\text{finite}} (n+d_i+a_ib_i-d_i-n)$$

which belongs to T. So, T is a normal subgroup of (N,+). If $\sum_{\text{finite}} (d_i+a_ib_i-d_i)$ is in T,

$$n \sum_{\text{finite}} (d_i+a_ib_i-d_i) = \sum_{\text{finite}} [nd_i+(na_i)b_i+n(-d_i)]$$

$$= \sum_{\text{finite}} (nd_i+(na_i)b_i- nd_i)$$

which belongs to T. So, T is an ideal in N. Now, T contains AB since T contains all products ab, a in A, b in B. Since any ideal in N, which contains all products ab, a in A, b in B, contains all elements in T, AB contains T. Thus, AB = T.

Let a be an arbitrary element of N.

Let $W = \{\pm(d+sa+ka-d) \mid s,\ d$ is in N, k an integer$\}$ and $V = \{\sum_{\text{finite}} x_i \mid x_i$ is in W$\}$. We will show that (a) = V.

Clearly V is a subset of (a). Now, V is a subgroup of (N,+). If $\sum_{\text{finite}} x_i$ is in V and n is in N,

$$n + (\sum_{\text{finite}} x_i) - n = n + \left[\sum_{\text{finite}} (x_i - n + n)\right] - n$$

$$= \sum_{\text{finite}} (n + x_i - n)$$

which belongs to V since $n + x_i - n$ belongs to W. So, V is a normal subgroup of (N,+). Since

$$n(d+sa+ka-d) = nd + n(sa) + n(ka) + n(-d)$$

$$= (nd+(ns)a+0a-nd) + (nd+(kn)a+0a-nd),$$

NV is a subset of V. Thus, V is an ideal in N. Since a belongs to V, (a) is a subset of V. Hence (a) = V.

Suppose that a and b belong to N. We show here that (ab) = (a)(b). Since ab belongs to (a)(b), (ab) is a subset of (a)(b). Let c, d, t, s belong to N and m, k be integers. Because of the right-distributive law, it is sufficient to show that (c+ta+ma-c)(d+sb+kb-d) belongs to (ab) to prove that (a)(b) is a subset of (ab). Now,

$$(c+ta+ma-c)(d+sb+kb-d) = (c+ta+ma-c)d + (c+ta+ma-c)sb + (c+ta+ma-c)kb - (c+ta+ma-c)d.$$

But, $(c+ta+ma-c)sb = csb + ts(ab) + ms(ab) - csb$

$$= (csb+ts(ab)-csb) + (cab+ms(ab)-csb)$$

and $(c+ta+ma-c)kb = c(kb) + (kt)ab + (mk)ab - c(kb)$

$$= (c(kb) + (kt)ab - c(kb)) + (c(kb) + (mk)ab - c(kb))$$

which belongs to (ab). Since (ab) is a normal subgroup of (N,+), (c+ta+ma-c)(d+sb+kb-d) belongs to (ab).
Hence, (a)(b) is a subset of (ab). Thus, (a)(b) = (ab).

We now verify that if P is an ideal in N, P is prime iff ab belongs to P implies a belongs to P or b belongs to P. Suppose P is prime, ab belongs to P, and a, b do not belong to P. By (i), (a)(b) is not a subset of P. So, (ab) is not a subset of P. But, since ab belongs to P, (ab) is a subset of P. This is a contradiction. So, a belongs to P or b belongs to P.

Now, suppose that if ab belongs to P, then a belongs to P or b belongs to P. Suppose P is not prime. Then, there exist ideals A and B such that A is not a subset of P, B is not a subset of P but AB is a subset of P. There exist a in A, b in B such that a, b do not belong to P. However, ab belongs to P. Hence, a belongs to P or b belongs

P. This is a contradiction. Thus, P is a prime ideal.

Ideals in R-rings

An element of a near-ring $(N,+,\cdot)$ is right distributive iff $(b+c)a = ba + ca$ for b, c in N. An element a is anti-right distributive iff $(b+c)a = ca + ba$ for b, c in N. An element a in N is weakly right distributive iff it is a finite sum of right and anti-right distributive elements. An R-ring is a near-ring in which every element is weakly right distributive (see [1]).

Suppose that N is an R-ring and the element a is right or anti-right distributive. Since

$$0a = (0+0)a = 0a + 0a,$$

$0a = 0$. A C-ring is a near-ring N in which $0a = 0$ for each a in N. In particular, an R-ring is a C-ring.

Suppose that J is an ideal in the R-ring N, j belongs to J and a belongs to N. Now, $a = a + a + \ldots + a_k$ where each a_i is right distributive or anti-right distributive. Since

$$ja = j(a_1+a_2+\ldots+a_k) = ja_1 + ja_2 + \ldots + ja_k$$

$$= ((0+j)a_1-0a_1) + ((0+j)a_2-0a_2) + \ldots + ((0+j)a_k-0a_k),$$

which is in J, JN is a subset of J.

Suppose that L is an additive normal subgroup such that NL and LN are subsets of L. Let l belong to L and a_1, a_2 belong to N. Now, $a_2 = b_1 + b_2 + \ldots + b_k$ where each b_i is right distributive or anti-right distributive. So

$$(a_1+l)a_2 - a_1a_2 = (a_1+l)(b_1+\ldots+b_k) - a_1(b_1+\ldots+b_k)$$

$$= (a_1+l)b_1 + (a_1+l)b_2 + \ldots + (a_1+l)b_k - a_1b_k - \ldots - a_1b_1.$$

Notice that $(a_1+l)b_k$ is either $a_1b_k + lb_k$ or $lb_k + a_1b_k$. In either case, $(a_1+l)b_k - a_1b_k$ belongs to L.

Let $(a_1+l)b_k - a_1b_k = l_k$. Hence

$$(a_1+l)a_2 - a_1a_2 = (a_1+l)b_1 + \ldots + (a_1+l)b_{k-1} + l_k - a_1b_{k-1} - \ldots - a_1b_1.$$

Again,

$$(a_1+l)b_{k-1} \text{ is either } a_1b_{k-1} + lb_{k-1} \text{ or } lb_{k-1} + a_1b_{k-1}.$$

If $(a_1+l)b_{k-1} = a_1b_{k-1} + lb_{k-1}$,

$$(a_1+l)b_{k-1} + l_k - a_1b_{k-1} = a_1b_{k-1} + lb_{k-1} + l_k - a_1b_{k-1}$$

$$= a_1b_{k-1} + l_1 - a_1b_{k-1}$$

for some l_1 in L. So, $(a_1+l)b_{k-1} + l_k - a_1b_{k-1}$ belongs to L.

If $(a_1+l)b_{k-1} = lb_{k-1} + a_1b_{k-1}$,

$(a_1+l)b_{k-1} + l_k - a_1b_{k-1} = lb_{k-1} + (a_1b_{k-1}+l_k-a_1b_{k-1}) = l_2 + l_3$ for some l_2, l_3 in L. So, $(a_1+1)b_{k-1} + l_k - a_1b_{k-1}$ is in L. We continue this finite process and find that $(a_1+1)a_2 - a_1a_2$ belongs to L. Thus, we see that an ideal in N is an additive normal subgroup J such that NJ and JN are subsets of J.

Ideals in C-rings

Suppose that $(N,+,\cdot)$ is a near-ring. We shall prove that N is a C-ring iff each ideal I is a normal subgroup such that NI and IN are subsets of I. First, suppose that N is a C-ring and that I is an ideal in N. Then,

$$(0+a)n - 0n = an$$

is in I if a belongs to I and n belongs to N. Hence, IN is a subset of I.

Now, suppose that each ideal I is a normal subgroup such that NI and IN are subsets of I. Consider the zero ideal $\{0\}$. Now, $\{0\}N$ is a subset of $\{0\}$. Thus $0n = 0$ for each n in N. Hence, N is a C-ring. Notice that the ideals in a C-ring are defined the same as ideals in a ring.

Principal ideals in near-rings with identities

Let $(N,+,\cdot)$ be a near-ring with multiplicative identity e. Take any element a in N.

Let

$$S_1 = \{\pm(n_4+n_3[(n_1+a)n_2-n_1n_2] - n_4) \mid n_1,\ n_2,\ n_3,\ n_4 \text{ are in } N\}$$

$$T_1 = \{\sum_{finite} x_i | x_i \text{ is in } S_1\}$$

$$S_2 = \{\pm(n_4+n_3[(n_1+y)n_2 - n_1n_2]-n_4) \left| \begin{matrix} n_1, n_2, n_3, n_4 \text{ are in } N \\ y \text{ is in } T_1 \end{matrix} \right.\}$$

$$T_2 = \{\sum_{finite} x_i | x_i \text{ is in } S_2\}$$

$$S_3 = \{\pm(n_4+n_3[(n_1+y)n_2 - n_1n_2]-n_4) \left| \begin{matrix} n_1, n_2, n_3, n_4 \text{ are in } N \\ y \text{ is in } T_2 \end{matrix} \right.\}$$

$$T_3 = \{\sum_{finite} x_i | x_i \text{ is in } S_3\}$$

. . . .

$$S_m = \{\pm(n_4+n_3[(n_1+y)n_2 - n_1n_2]-n_4) \left| \begin{matrix} n_1, n_2, n_3, n_4 \text{ are in } N \\ y \text{ is in } T_{m-1} \end{matrix} \right.\}$$

$$T_m = \{\sum_{finite} x_i | x_i \text{ is in } S_m\}$$

Notice that S_m is a subset of T_m, which is a subset of S_{m+1}. Let H be the union of the T_i, i = 1, 2, ... We will show that H = (a).

Take h_1, h_2 in H. Now h_1, h_2 are in S_m for some positive integer m. So, $h_1 - h_2$ is in T_m. Hence, $h_1 - h_2$ is in H. So H is an additive subgroup of (N,+). If h belongs to H and c belongs to N, $c + h - c = c + \left[\sum_{finite} x_i\right] - c$ where each x_i is in S_m for some positive integer m. Now, $c + \left[\sum_{finite} x_i\right] - c = c + \left[\sum_{finite} (x_i - c + c)\right] - c$, which is in T_m. So, H is a normal subgroup of (N,+). Clearly, NH is a subset of H. Now, $(n_1+h)n_2 - n_1n_2$ is in S_{m+1} for n_1, n_2 in N.

Thus, H is an ideal in N. Since

$$e[(-a+a)e - (-a)e] = e[0e + a] = ea = a,$$

a belongs to H. Hence, H contains (a). But, any ideal which contains a contains all elements of H. So, (a) contains H. Thus, (a) = H.

If N is a ring, $H = \{\sum_{finite} n_{1_i} a n_{2_i} \mid n_{1_i}, n_{2_i} \text{ are in } N\}$.

Suppose that A and B are two ideals in N. If we let

$$S_1 = \{\pm(n_4+n_3[(n_1+ab)n_2 - n_1n_2]-n_4 \mid {n_1, n_2, n_3, n_4 \text{ are in } N, \atop a \text{ is in } A,\ b \text{ is in } B}\},$$

then H = AB.

CHAPTER III

IDEALS IN MATRIX NEAR-RINGS

Ideals in matrix rings

Suppose that R is a ring. Let R_m denote the set of all matrices of order m over R. We will use the notation (a_{ij}) to denote an arbitrary element of R_m. We define addition and multiplication on R_m by

$$(a_{ij}) + (b_{ij}) = (c_{ij})$$

and $$(a_{ij})(b_{ij}) = (d_{ij})$$

where $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$.

Hence, R_m is a ring. Moreover if I is an ideal in R then the set I_m of all matrices of order m over I is an ideal in R_m (see [4]).

Ideals in matrix near-rings

Suppose that N is a near-ring and let N_m denote the set of all matrices of order m over N. Let addition and multiplication be defined on N_m as in the case of rings. We will give an example to show that N_m is not necessarily a near-ring.

Let N be the first near-ring defined on page 9. Consider N_2. Let

$$(a_{ij}) = \binom{22}{00},\ (b_{ij}) = \binom{40}{50},\ \text{and}\ (c_{ij}) = \binom{40}{00}.$$

Now

$$(a_{11}b_{11}+a_{12}b_{21})c_{11} + (a_{11}b_{12}+a_{12}b_{22})c_{21} = (2\cdot 4+2\cdot 5)4 + (2\cdot 0+2\cdot 0)0$$

$$= (4+5)4 = 3\cdot 4 = 0$$

is the element in the first row, first column of the matrix $[(a_{ij})(b_{ij})](c_{ij})$.
But

$$a_{11}(b_{11}c_{11}+b_{12}c_{21}) + a_{12}(b_{21}c_{11}+b_{22}c_{21}) = 2\cdot 4\cdot 4 + 2\cdot 5\cdot 4$$

$$= 4 + 4 = 2$$

is the element in the first row, first column of the matrix $(a_{ij})[(b_{ij})(c_{ij})]$. So the associative law for multiplication does not hold in N_m. Notice that (N,+) is abelian. Thus, even if (N,+) is abelian, N_m is not necessarily a near-ring.

We wish to find conditions on N so that N_m is a near-ring. Now, N may not have a multiplicative identity. However, from [1], we know that N can be embedded into a near-ring with a multiplicative identity. So let us assume throughout the rest of the chapter that N has an identity e. We can now show that N_m satisfies the left distributive law iff (N,+) is abelian.

Suppose that N_m satisfies the left distributive law. Let x_1, x_2 belong to N. Now,

$$\begin{bmatrix} e & e & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ x_1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} e & e & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Also,

$$\begin{bmatrix} e & e & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\left[\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ x_1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} x_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} e & e & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_2 & 0 & 0 & \cdots & 0 \\ x_1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_2+x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, (N,+) is abelian. If (N,+) is abelian, the left distributive law in N_m follows easily from the left distributive law in N.

We will assume that (N,+) is abelian. We now prove that N_m is a near-ring iff N is a ring. First, suppose that N_m is a near-ring and that x, y, z belong to N. Now,

$$\left[\begin{pmatrix} e & e & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x & 0 & \cdots & 0 \\ y & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right] \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (x+y)z & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Also

$$\begin{pmatrix} e & e & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \left[\begin{pmatrix} x & 0 & \cdots & 0 \\ y & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} xz+yz & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Hence, $(x+y)z = xz + yz$. So, N is a ring. If N is a ring, N_m is a ring and hence a near-ring. Thus, if N_m is a near-ring, every ideal in N_m is of the form I_m where I is an ideal in N (see [4]).

Let us return to the case where N is an arbitrary near-ring. Assume that N_m is also a near-ring. Then N will not necessarily satisfy the right distributive law. We will give an example to verify this statement. Let C_4 be the cyclic group of order 4. From [2], $N = (C_4,+,\cdot)$ is a near-ring under the following multiplication:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	2	0	2
2	0	0	0	0
3	0	0	0	0

Now, $(3+1)3 = 0\cdot 3 = 0$. But, $3\cdot 3 + 1\cdot 3 = 0 + 2 = 2$. So N does not satisfy the right distributive law. However, $(ab+cd)e = abe + cde$ for arbitrary elements a, b, c, d, e in N. Hence, the associative law for multiplication holds in N_m, where m is any positive integer. So, N_m is a near-ring. However, if N is a near-ring, N_m is a near-ring and I is an ideal in N, then I_m is an ideal in N_m. The proof of this statement is trivial.

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