FROM GENERALIZED FOURIER TRANSFORMS TO COUPLED SUPERSYMMETRY

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
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FROM GENERALIZED FOURIER TRANSFORMS TO COUPLED SUPERSYMMETRY

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Abstract

The Fourier transform, the quantum mechanical harmonic oscillator, and supersymmetric quantum mechanics are well-studied objects in mathematics. The relations between them are also well-understood. The traditional understanding of each appears to suggest rigid structures that do not leave much flexibility for alternatives. In this thesis, we will develop generalizations of the Fourier transform and explore their relations. Exploring the analytic structure of these integral transforms, particularly their related Hamiltonians, naturally leads to Hamiltonian-like operators which have analytic and algebraic structures.

The algebraic structure underlying the Hamiltonian-like operators associated to generalized Fourier transforms suggests a much-more-general abstract formulation. To this end, we introduce the coupled supersymmetry (coupled SUSY) algebraic framework which unifies the quantum mechanical harmonic oscillator and supersymmetric quantum mechanics in a more complete way. The coupled SUSY framework subsumes the quantum mechanical harmonic oscillator and provides a broader class of systems which have interesting functional analytic and algebraic structures. In this setting, one is able to develop further generalizations of the Fourier transform. A further generalization of coupled SUSY is briefly presented which appears to offer new insights into supersymmetry.

Contents

1 Introduction **1**

2 Fourier Transforms **5**

2.1 Fourier Series 5

2.2 The Fourier Transform 7

3 Generalized Fourier Transforms **15**

3.1 Generalizing the Fourier Transform 15

3.2 A Family of Integral Transforms 17

3.3 The Φ_n transform and its domain 25

3.4 A Riemann-Lebesgue Lemma for Φ_n 26

3.5 Some eigenfunctions of Φ_n 28

3.6 Φ_n as an L^2 Isometry 32

3.7 The Spectrum of Φ_n 34

3.8 The Spectral Analysis of $\mathscr{D}_n$ 35

3.9 The Φ_n and Fourier-Bessel transforms 36

3.10 The Short-Time Φ_n Transform 38

3.11 The Uncertainty Principle for Φ_n 43

3.12 A Family of Related Integral Transforms 45

4 Supersymmetric Quantum Mechanics 48

4.1 The Quantum Mechanical Harmonic Oscillator 48

4.2 Supersymmetric Quantum Mechanics 56

5 Coupled Supersymmetric Quantum Mechanics and Associated Integral Operators 61

5.1 Coupled Supersymmetry and Its Lie Algebra 62

5.2 An Energy Ladder Structure for Coupled SUSY 69

5.3 Coupled SUSY and Other Oscillator Systems 74

5.4 Coherent States for Coupled SUSY Systems 76

5.5 Uncertainty Principles for Coupled SUSY Systems 80

5.6 A Family of Coupled SUSY Integral Transforms 86

5.7 The Coupled SUSY Algebra 89

5.8 Future Work: A Generalization of Coupled SUSY 93

Bibliography 97

Chapter 1

Introduction

The Fourier transform plays an integral role in mathematics and science. It serves as the cornerstone of abstract harmonic analysis and signal processing. It is quite appealing for many reasons, namely it is a unitary operator on $L^2(\mathbb{R})$ that is closely connected to the group structure of the real line and converts differential operators into multiplication operators. The Fourier transform is extremely important in quantum mechanics. In the context of quantum mechanics, the Fourier transform takes a wavefunction ψ expressed in the spatial domain and then expresses it in the momentum domain and vice versa. Physically, it can be thought of as extracting the momentum information of the wavefunction. The Fourier transform also gives a rigorous mathematical justification for the Heisenberg uncertainty principle.

The Fourier transform is also critical in the spectral theory of the (negative) Laplacian: $-\Delta$. When viewed on an appropriate dense subspace of $L^2(\mathbb{R})$, this operator is unbounded, essentially self-adjoint, and positive. The spectral theorem for unbounded operators says that a unitary operator $\mathcal{U}$ exists such that $-\Delta$ is unitarily equivalent to multiplication by a nonnegative function under $\mathcal{U}$. Analysis shows that

$\mathcal{U}$ is exactly the Fourier transform. In one real variable, $-\frac{d^2}{dx^2}$ has an image of y^2 under the Fourier transform. (In higher dimensions, $-\sum_i \frac{d^2}{dx_i^2}$ has image $\sum_i x_i^2$.) Likewise, the multiplication operator x^2 has an image of $-\frac{d^2}{dy^2}$ under the Fourier transform. As such, the operator $-\frac{d^2}{dx^2}+x^2$ will be form-invariant under the Fourier transform. This can be readily realized as the quantum-mechanical harmonic oscillator Hamiltonian.

The purpose of this thesis is to explore integral transforms which generalize the Fourier transform and develop Hamiltonian-like objects which generalize the quantum-mechanical harmonic oscillator Hamiltonian. This pursuit leads to a rich, general theory when coupled with the language of supersymmetric quantum mechanics. This new theory is denoted as coupled supersymmetry, or coupled SUSY. In this framework, one can define pairs of integral transforms that generalize the Fourier transform which are closely related. In the context of the quantum-mechanical harmonic oscillator, these structures collapse into pairs of duplicates, and so these structures are hidden in this case.

Generalizing the Fourier transform on $\mathbb{R}$ and the quantum-mechanical harmonic oscillator are hardly new ideas. The Hankel, or Fourier-Bessel, transform [28, 24, 25] and the fractional Fourier transform [33] are two such generalizations of the Fourier transform. The Fourier-Bessel transform is very closely related to the work contained herein. There has been some work characterizing and studying general integral transforms on $\mathbb{R}$ [11, 12] and some work on periodic integral transforms [52, 32]. There is also a generalization of the Fourier transform to arbitrary locally compact abelian groups which subsumes both Fourier series and Fourier transforms on $\mathbb{R}$. This generalization of the Fourier transform is quite general and very satisfactory, though it does not in general capture a differential structure. This generalization is founded on continuous homomorphisms of the locally compact abelian group to the unit circle $\mathbb{T}$.

Such continuous homomorphisms serve as the integral kernel for the Fourier transform. It is a simple exercise to show that in the case of the group of real numbers under addition, the Fourier kernel is effectively unique and is exactly $\exp(-ixy)$. Furthermore, Fourier transforms come in pairs in this setting: one may define a Fourier transform on a group G and its dual group $\widehat{G}$. A generalization closely related to the abstract harmonic analytic Fourier transform is the Dunkl transform [30, 42].

Supersymmetric quantum mechanics is a broad generalization of the ladder operator formalism of the quantum-mechanical harmonic oscillator. The connection between the quantum-mechanical harmonic oscillator and the Fourier transform can be made explicit beyond the connection via the Gaussian as an eigenfunction of each. The creation and annihilation operators $x \pm ip$ in the quantum-mechanical harmonic oscillator are *eigenoperators* [32] of the Fourier transform, i.e., they are form-invariant under the Fourier transform. This parallel between the quantum-mechanical harmonic oscillator and the Fourier transform suggests that, at least in special subcases of supersymmetric quantum mechanics, Fourier-like transforms exist and have the supersymmetric charge operators as eigenoperators.

This thesis has two main sections. The first section comprises Chapters 2 and 3 and focuses on integral transform theory. Chapter 2 contains a brief overview of Fourier series and transform theory. Chapter 3 contains generalizations of the Fourier transform and explores uncertainty relations and a notion of duality between integral transforms. The analysis of the generalized Fourier transform leads to the consideration of Hamiltonian-like operators which have a supersymmetric nature. This leads naturally into the second section of the thesis. The second section comprises Chapters 4 and 5. Chapter 4 contains a brief review of the quantum-mechanical harmonic oscillator and supersymmetric quantum mechanics. Chapter 5 introduces coupled supersymmetry and explores its analytic and algebraic structures.

The work in Chapter 3 has been accepted to the *Journal of Fourier Analysis and Applications* [54] and is to appear. The work in Chapter 5 has been submitted to arXiv [55].

Chapter 2

Fourier Transforms

2.1 Fourier Series

The Fourier transform serves as the basis of much of the work in this thesis, due in no small part to its many nice properties. The Fourier transform is, in some sense, a continuum analogue of the Fourier series [8, p. 92] and so we choose to briefly investigate the nature of Fourier series. The Fourier series of a sufficiently nice compactly supported (or periodic) function extracts the amplitude and frequency information by expanding the function as a (possibly infinite) linear combination of sines and cosines. Historically, Fourier series were developed to approximate general solutions to the heat equation on a bounded domain [20, 21].

Fourier series play an integral role in communications, largely due to Shannon's sampling theorem [47] which states that a band-limited signal can be completely reconstructed from a sufficient number of discrete samples. Much real-world data are continuous, however only discrete data can be stored or analyzed on computers, making the sampling theorem a very crucial theorem in signal reconstruction. The sampling theorem guarantees that the reconstructed data agrees with the original signal within a given tolerance. To this end, Fourier series are quite important in real-world applications. We now formally define the Fourier series.

Given a function $f \in L^1([0, 2\pi])$, we may define its (formal) Fourier series (provided that it exists) as the function

$$\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{inx} \mathcal{F}f(n), \tag{2.1}$$

where $x \in [0, 2\pi]$ and

$$\mathcal{F}f(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-iny} f(y)\, dy. \tag{2.2}$$

Indeed this is the largest domain for which Fourier series can be defined since

$$\int_0^{2\pi} |e^{-iny} f(y)|\, dy = \int_0^{2\pi} |f(y)|\, dy.$$

Both integrals converge if and only if $f \in L^1([0, 2\pi])$. Note that the L^p spaces are nested on $[a, b]$ where $-\infty < a < b < \infty$. (In general, if $\mathfrak{X}$ is a finite measure space and $p_2 > p_1$, then $L^{p_2}(\mathfrak{X}) \subset L^{p_1}(\mathfrak{X})$ [19, p. 186].)

Remark 2.1. *The integrals that appear herein are to be interpreted as Lebesgue integrals. However, in many situations of interest in this thesis, the Lebesgue integral may be treated as a Riemann integral.*

Carleson's theorem [10, 29] shows that for $f \in L^p([0, 2\pi])$ with $1 < p < \infty$, then the Fourier series of f converges almost everywhere to f when the Fourier series is summed as a principal value sum, that is to say that for $f \in L^p([0,2\pi])$,

$$f(x) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \mathcal{F}f(n)e^{inx} \tag{2.3}$$

for almost every $x \in [0,2\pi]$.

2.2 The Fourier Transform

An inherent fault in the theory of Fourier series is that it is built on functions defined on bounded intervals, or more generally on periodic functions. Many functions that are of general interest are not defined on bounded intervals and are not periodic, e.g., $\exp(-x^2/2)$ for $x \in \mathbb{R}$. This leads to the theory of the Fourier transform. In some sense, the Fourier transform may be viewed as a continuum analogue of Fourier series.

The Fourier transform can be abstractly defined via group theoretic techniques [18, 44, 48]. Given a locally compact abelian group G, one may define the set of *continuous* characters, denoted $\widehat{G}$, to be the group of functions $\chi : G \to \mathbb{T}$, where $\mathbb{T}$ is the unit circle, satisfying $\chi(g_1g_2) = \chi(g_1)\chi(g_2)$. The set of continuous characters forms a group under multiplication called the *dual group*. One can define the Fourier transform from G to $\widehat{G}$ by $\mathcal{F}f(\chi) = \int_G \overline{\chi(g)} f(g)\, dg$ and a Fourier transform from $\widehat{G}$ back to G similarly. These Fourier transforms are in some sense inverses of each other. Particularly, for suitably nice functions, the Fourier transforms are indeed inverses of each other. Notably, Fourier transforms come in pairs in the context of locally compact groups. For more information see [18, 44, 48]. We will see in the next chapter that our generalizations of the Fourier transform also come in pairs.

It is a simple exercise to show that the dual group of the group $[0, 2\pi)$ under addition modulo 2π is the group of integers [18, p. 90]. In the defintion of the Fourier series appeared an integral over $[0, 2\pi)$ and a sum over $\mathbb{Z}$. Hence, the Fourier series is a specific realization of the abstract Fourier transform on $[0, 2\pi)$ and $\mathbb{Z}$. Considering instead the group of real numbers $\mathbb{R}$ with addition, a straightforward argument shows that if $f : \mathbb{R} \to \mathbb{T}$ is a continuous character, then $f(x) = \exp(ixy)$ for some fixed $y \in \mathbb{R}$ [18, p. 90]. Thus, the dual group of $\mathbb{R}$ is again $\mathbb{R}$, and so we expect that the Fourier and inverse Fourier transforms on $\mathbb{R}$ are closely related. This leads into our first definition.

Definition 2.2. Let $f \in L^1(\mathbb{R})$ and $y \in \mathbb{R}$, then its *Fourier transform*, denoted $\mathcal{F}f$, is defined to be

$$\mathcal{F}f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)\, dx. \tag{2.4}$$

This is well-defined for $f \in L^1(\mathbb{R})$ since

$$\int_{-\infty}^{\infty} |e^{-ixy} f(x)|\, dx = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Both integrals $\int_{-\infty}^{\infty} e^{-ixy} f(x)\, dx$ converge for $y \in \mathbb{R}$ as it is absolutely integrable. Furthermore, this is the largest domain of definition for the Fourier transform as an integral operator.

The Fourier transform has numerous properties which are direct consequences of the Fourier kernel $\exp(-ixy)$ being a continuous character on $\mathbb{R}$, the Lebesgue measure, and the $L^1(\mathbb{R})$ space. Some of these properties are summarized below. Proofs may be found in [43, p. 179].

Theorem 2.3. *Let $f, g \in L^1(\mathbb{R})$, $x', y, y' \in \mathbb{R}$, $\alpha \in \mathbb{R}^*$, $f_{x'}(x) = f(x - x')$, $(f * g)(x) = \int_{-\infty}^{\infty} f(x - \widetilde{x}) g(\widetilde{x})\, d\widetilde{x}$, and $\mathscr{D}_\alpha f(x) = \sqrt{|\alpha|} f(\alpha x)$, then*

1. *$\mathcal{F}f \in C_0(\mathbb{R})$, i.e., $\mathcal{F}f$ is continuous on $\mathbb{R}$ and goes to zero at infinity,*

2. *$\mathcal{F}(f_{x'})(y) = e^{-ix'y}\mathcal{F}f(y)$,*

3. *$\mathcal{F}(e^{ixy'}f)(y) = \mathcal{F}f(y - y')$,*

4. *$\mathcal{F}(f * g)(y) = \sqrt{2\pi}\mathcal{F}f(y) \cdot \mathcal{F}g(y)$,*

5. *$\mathcal{F}(\mathscr{D}_\alpha f)(y) = \mathscr{D}_{\alpha^{-1}}\mathcal{F}f(y)$,*

6. *If $f \in AC(\mathbb{R})$, then $\mathcal{F}(f')(y) = iy\mathcal{F}f(y)$,*

7. *If $xf \in L^1(\mathbb{R})$, then $\mathcal{F}(xf)(y) = -i\frac{d}{dy}\mathcal{F}f(y)$.*

The first property is known as the Riemann-Lebesgue lemma. The second and third properties are restatements of the fact that the function $x, y \mapsto \exp(-ixy)$ is a group homomorphism of $\mathbb{R}$ to $\mathbb{T}$. The fourth property relies on this fact, in addition to the Fubini-Tonelli theorem. The fifth is a consequence of the fact that the Fourier kernel is a product kernel, i.e., $f(x, y) = g(xy)$ for some function g. The last two are consequences of the fact that the function $x \mapsto \exp(\lambda x)$ is an eigenfunction of the derivative operator.

Definition 2.4. Let $f \in L^1(\mathbb{R})$ and $y \in \mathbb{R}$. The inverse Fourier transform of f, denoted $\mathcal{F}^{-1}f$, is defined to be

$$\mathcal{F}^{-1}f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx. \tag{2.5}$$

Indeed, as the name suggests, the operator $\mathcal{F}^{-1}$ does act as the inverse of the Fourier transform for such functions f. Comparing this to the definition of the Fourier series, we see that $\mathcal{F}f$ plays an analogous role to the integral $(2\pi)^{-1/2}\int_0^{2\pi} e^{-iny} f(x)\, dx$, whereas $\mathcal{F}^{-1}$ plays the role of the summation and the following identity holds for almost every $x \in \mathbb{R}$ provided that $f, \mathcal{F}f \in L^1(\mathbb{R})$ [43, p. 185]:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \mathcal{F}f(x)\, dx. \tag{2.6}$$

If $f \in L^1(\mathbb{R})$ it is not guaranteed that $\mathcal{F}f \in L^1(\mathbb{R})$ and so the above integral may not exist, i.e., f may not be invertible. A typical example of such an f is $f = \frac{1}{2}\chi_{[-1,1]} \in L^1(\mathbb{R})$. For this choice of f, $\mathcal{F}f(y) = \sqrt{\frac{2}{\pi}}\frac{\sin y}{y}$. $\mathcal{F}f$ is not absolutely integrable on $\mathbb{R}$ and is therefore not in $L^1(\mathbb{R})$. Instead one may argue that if $\mathcal{F}f \in L^1(\mathbb{R})$, then we could realize $\mathcal{F}^{-1}\mathcal{F}f(y)$ as $\mathcal{F}^2 f(-y)$. $\mathcal{F}^2 f(-y)$ will be continuous by the Riemann-Lebesgue lemma, however the original definition of f is not continuous. An adaptation of the latter argument works quite well for more general locally compact abelian groups.

The L^1 theory of the Fourier transform is quite nice, but it is perhaps the L^2 theory that is most appealing as its functional analytic properties on $L^2(\mathbb{R})$ are even richer. Many of the L^1 properties can be adapted to the L^2 theory. To develop the Fourier transform on $L^2(\mathbb{R})$, one first considers the dense subspace $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ in $L^2(\mathbb{R})$. That this is a dense subspace of $L^2(\mathbb{R})$ is fairly straightforward. Given $f \in L^2(\mathbb{R})$, define the sequence $f_n = \chi_{[-n,n]} f$. These functions are compactly supported and in $L^2(\mathbb{R})$. By the nesting of L^p spaces on finite measure spaces [19, p. 186] (of which $[-n, n]$ under the Lebesgue measure is an example), each of the f_n are in $L^1(\mathbb{R})$. A straightforward calculation shows that $\|f - f_n\|_2 \to 0$ as n tends to infinity. The heuristic argument is that in order for $f \in L^2(\mathbb{R})$, the integral of its tail, represented by $f - f_n$, must go to zero.

The Fourier transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an L^2 isometry. That is to say that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\mathcal{F}f \in L^2(\mathbb{R})$ and $\|\mathcal{F}f\|_2 = \|f\|_2$. The standard proof for this involves regularizing the integrals with exponentially decaying functions [43], be they of the form $\exp(-|t|)$ or $\exp(-x^2)$. Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and the Fourier transform is an L^2 isometry from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, it extends to an isometry on $L^2(\mathbb{R})$. This extension cannot be realized as an integral operator directly as the integral of $\exp(-ixy)$ against an L^2 function need not exist.

The extension of the Fourier transform from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to all of $L^2(\mathbb{R})$ is traditionally also called the Fourier transform, though the understanding is not lost as the Fourier transforms on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and on $L^2(\mathbb{R})$ are quite different in nature. The Fourier transform on $L^2(\mathbb{R})$ has numerous properties, sharing many with the Fourier transform on $L^1(\mathbb{R})$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $L^2(\mathbb{R})$. Some unique properties of the $L^2(\mathbb{R})$ Fourier transform are

1. $\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle$,
2. $\mathcal{F}$ is onto $L^2(\mathbb{R})$,
3. $\mathcal{F}$ is a unitary,
4. $\mathcal{F}^4 = I$,
5. $\sigma(\mathcal{F}) = \{\pm 1, \pm i\}$.

That $\mathcal{F}^4 = I$ follows from the fact that $\mathcal{F}^2$ acts as a reflection operator, i.e., $\mathcal{F}^2 f(x) = f(-x)$ on $L^2(\mathbb{R})$. This is ultimately a consequence of the symmetry between the Fourier and inverse Fourier transforms, particularly, the two differ only insofar that $\exp(-ixy)$ and $\exp(ixy)$ differ by an overall sign change. Perhaps the most surprising and enticing property of the $L^2(\mathbb{R})$ Fourier transform is the uncertainty principle associated to it.

Theorem 2.5. *Let $f \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$ be such that $x \mapsto xf(x) \in L^2(\mathbb{R})$ and $y \mapsto y\mathcal{F}f(y) \in L^2(\mathbb{R})$, then*

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} y^2|\mathcal{F}f(y)|^2\,dy\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^2. \tag{2.7}$$

There are many proofs for this theorem [22, 26, 49], most of which make use of some form of the Cauchy-Schwarz inequality. We provide one proof below.

Proof. Let $f \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$ be such that $x \mapsto xf(x) \in L^2(\mathbb{R})$ and $y \mapsto y\mathcal{F}f(y) \in L^2(\mathbb{R})$, then f' exists and $f' \in L^2(\mathbb{R})$. By integration by parts,

$$\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^2 = 4\left(\mathrm{Re}\int_{-\infty}^{\infty} xf(x)f'(x)\,dx\right)^2.$$

Note that f goes to zero at infinity since $f \in AC(\mathbb{R})$ and $|f|^2 \in L^1(\mathbb{R})$. Using Cauchy-Schwarz with the auxiliary functions $\eta(x) = xf(x)$ and $\xi(x) = f'(x)$ as well as the established equivalence $\mathcal{F}(f')(y) = iy\mathcal{F}f(y)$,

$$\begin{aligned}
4\left(\mathrm{Re}\int_{-\infty}^{\infty} \eta(x)\xi(x)\,dx\right)^2 &\leq 4\left(\int_{-\infty}^{\infty} |\eta(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} |\xi(x)|^2\,dx\right)\\
&= 4\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} y^2|\mathcal{F}f(y)|^2\,dy\right).
\end{aligned}$$

Thus

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} y^2|\mathcal{F}f(y)|^2\,dy\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^2.$$

Equality holds above if and only if $\xi = -\lambda\eta$ for some $\lambda \in \mathbb{C}$, i.e., $f'(x) = -\lambda xf(x)$ and so $f(x) = \exp(-\lambda x^2/2)$. Provided that $\mathrm{Re}(\lambda) > 0$, $f \in L^2(\mathbb{R})$ and results in a modulated Gaussian. A simple computation shows that for such a choice of f, equality indeed holds. $\square$

The function $\exp(-x^2/2)$ is not just a minimizer for the uncertainty product, it is also an eigenfunction of the Fourier transform. We prove this below. Define f by

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} e^{-\frac{x^2}{2}}\, dx. \tag{2.8}$$

Taking a derivative of both sides with respect to y and noting that we may differentiate under the integral sign, we get

$$f'(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ixe^{-ixy} e^{-\frac{x^2}{2}}\, dx.$$

$-x\exp(-x^2/2)$ can readily be viewed as the derivative of $\exp(-x^2/2)$, and an integration by parts yields

$$f'(y) = -\frac{y}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} e^{-\frac{x^2}{2}}\, dx = -yf(y).$$

The solution to this is clearly of the form $C\exp(-x^2/2)$.

Another standard technique for showing that the Gaussian is an eigenfunction of the Fourier transform makes use of rewriting $\exp(-ixy)\exp(-x^2/2)$ as a single exponential with complex argument, completing the square, and making use of the entirety of the exponential function to do a complex change of variables.

As noted above, the Fourier transform interchanges the roles of the multiplication operator $\mathcal{M}_{ix}$ and the derivative operator $\frac{d}{dy}$. The Fourier transform also interchanges the roles of $\mathcal{M}_y$ and $i\frac{d}{dx}$; as such the operator $\frac{d}{dx}+x$ is form-invariant under the Fourier transform, that is to say that

$$\mathcal{F}\left(\left(\frac{d}{dx}+x\right)f\right)(y)=i\left(\frac{d}{dy}+y\right)\mathcal{F}f(y). \tag{2.9}$$

Likewise,

$$\mathcal{F}\left(\left(-\frac{d}{dx}+x\right)f\right)(y)=-i\left(-\frac{d}{dy}+y\right)\mathcal{F}f(y). \tag{2.10}$$

The operators $\frac{d}{dx}+x$ and $-\frac{d}{dx}+x$ will play an important role in Chapter 3.

The Fourier transform is, unsurprisingly, critical to the spectral theory of the operator $-\frac{d^2}{dx^2}$. This should be expected since the Fourier transform is unitary and also enjoys the property that its kernel $\exp(-ixy)$ is a (non-L^2) eigenfunction of the operator $-\frac{d^2}{dx^2}$. The image of the operator $-\frac{d^2}{dx^2}$ under the Fourier transform is y^2, and likewise the image of the multiplication operator $\mathcal{M}_{x^2}$ under the Fourier transform is the operator $-\frac{d^2}{dy^2}$. As such, it is natural to construct the operator

$$-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2.$$

This operator will be invariant under the Fourier transform, much like the operators $\frac{d}{dx}+x$ and $-\frac{d}{dx}+x$. Such operators are called eigenoperators and some theory has been worked out for them [32].

Chapter 3

Generalized Fourier Transforms

3.1 Generalizing the Fourier Transform

Through its connection with the group structure of the real numbers, the Fourier transform appears to leave little flexibility in its design. Part of the popularity of wavelets in harmonic analysis can be attributed to the variety of ways that scaling functions and associated wavelets, building blocks for signal analysis, can be chosen for different purposes in time-frequency analysis. A structural difference between Fourier and wavelet analysis is the use of dilations in the definition of the wavelet transform which are related to the affine group.

This section follows a path to the Fourier transform that orients itself closely with the structure of wavelets, making the intertwining relationship with dilations a defining property rather than a relationship between translations and modulations. If one had to choose a "scaling function" associated with the Fourier transform, it would arguably be the Gaussian; a suitably chosen Gaussian is an eigenfunction of the Fourier transform, and its translates and modulations are related by analytic continuation. The Gaussian is also an uncertainty minimizer, a fact that has relevance for the short-time Fourier transform, which extracts local frequency information by modulating with a moving window and subsequently applying the Fourier transform.

The Gaussian also plays the role of a low-pass filter in some applications; however, while it is localized in both time and frequency, it is not considered to be close to an ideal low-pass filter [7]. Particularly, the Gaussian decays rapidly to zero, whereas a nearly ideal low-pass filter should be nearly 1 over an interval of interest and decay rapidly to zero outside of it. On the other hand, because of the nearly discontinuous behavior of an approximately ideal filter, one would not expect its Fourier transform to be well-localized which is desirable from a numerical point of view. The simplest way in which to address this deficiency is to find an integral transform with an eigenfunction possessing simultaneous localization in both domains, something that seems impossible when considering the various manifestations of the uncertainty principle in harmonic analysis.

In this chapter, we will establish that there is a family of integral transforms $\{\Phi_n\}_{n=1}^{\infty}$, each Φ_n densely defined on $L^2(\mathbb{R})$, which generalize the properties of the Fourier transform in the following way:

1. If $g_n(x) = e^{-\frac{x^{2n}}{2n}}$, $n \in \mathbb{N}$, then $\Phi_n g_n = g_n$.
2. If $\alpha \in \mathbb{R}\backslash\{0\}$ and $\mathcal{D}_\alpha$ is the dilation operator given above, then $\Phi_n \mathcal{D}_\alpha = \mathcal{D}_{\alpha^{-1}} \Phi_n$.
3. The operator Φ_n is unitary and can be defined as an integral transform when its domain is suitably restricted to a dense set in $L^2(\mathbb{R})$.
4. $\Phi_n^4 = I$ and its eigenvalues and spectrum are comprised only of $\pm 1, \pm i$.

Moreover, an uncertainty principle, some spectral analysis of an unbounded Laplacian will be explored, a short-time analogue of Φ_n will be developed, and a family of integral transforms closely related to Φ_n will be established.

The Gaussian is a special case of the family of Gaussian-like functions $\{g_n\}_{n=1}^{\infty}$ featured in property (1). The guiding principle is to retain as many properties of the Fourier transform as possible while demanding that Φ_n leaves g_n invariant. Based on these axioms, we derive that Φ_n is defined as an integral transform,

$$\Phi_n g(y) = \int_{-\infty}^{\infty} \varphi_n(xy) g(x) \, dx, \quad \text{a.e. } y \in \mathbb{R},$$

for each sufficiently regular function g with the integral kernel $\varphi_n(xy) = c_n(xy) + is_n(xy)$,

$$c_n(\eta) = \frac{1}{2} |\eta|^{n-\frac{1}{2}} J_{-1+\frac{1}{2n}} \left(\frac{|\eta|^n}{n} \right),$$

and

$$s_n(\eta) = -\frac{1}{2} \operatorname{sgn}(\eta) |\eta|^{n-\frac{1}{2}} J_{1-\frac{1}{2n}} \left(\frac{|\eta|^n}{n} \right).$$

The Fourier transform emerges as the special case $\mathcal{F} = \Phi_1$. The functions c_n and s_n are shown to be solutions to the eigenvalue equation of a (singular) Laplacian, $-\frac{d}{d\eta} \frac{1}{\eta^{2n-2}} \frac{d}{d\eta} g(\lambda\eta) = \lambda^{2n} g(\lambda\eta)$. This shows that the asymptotic oscillatory behavior of the kernel can be tuned by choosing n, which is expected to be useful in applications of signal analysis to functions with chirp-like components.

3.2 A Family of Integral Transforms

We first want to show that a densely defined, bounded integral operator on $L^2(\mathbb{R})$ satisfying the dilation property given in (2) has an integral kernel, φ, of the form

$\varphi(y,t) = f(xy)$ for some function f.

Proposition 3.1. *Suppose $\mathcal{T}$ is a bounded integral operator defined on a dense subspace $\mathfrak{X}$ of $L^2(\mathbb{R})$ such that $\mathfrak{X}$ is invariant under each $\mathcal{D}_\alpha$, $\alpha \neq 0$, and $\mathcal{T}\mathcal{D}_\alpha g = \mathcal{D}_{\alpha^{-1}}\mathcal{T} g$ for all $g \in \mathfrak{X}$. If φ is a kernel for $\mathcal{T}$, then we can choose φ to be of the form $\varphi(y,t) = f(xy)$ for some function f.*

Proof. Suppose $g \in \mathfrak{X}$ and let $\alpha \in \mathbb{R} \setminus \{0\}$. $\mathcal{T}\mathcal{D}_\alpha g$ and $\mathcal{D}_{\alpha^{-1}}\mathcal{T} g$ are then defined almost everywhere. From the dilation property and by a change of variables, we have

$$\int_{-\infty}^{\infty} \varphi(\alpha x, y) g(x)\, dx = \int_{-\infty}^{\infty} \varphi(x, \alpha y) g(x)\, dx, \quad \text{a.e. } y \in \mathbb{R}. \tag{3.1}$$

This identity also holds almost everywhere when choosing g among a countable, dense subset of $\mathfrak{X}$, which is dense in $L^2(\mathbb{R})$, and thus for a fixed $\alpha \neq 0$, $\varphi(\alpha x, y) = \varphi(x, \alpha y)$ for almost every $x, y \in \mathbb{R}$. Next, this identity is valid when selecting α from a countable set, say the rationals. Scaling then gives $\varphi(x\alpha^{-1}, \alpha y) = \varphi(x, y)$ almost everywhere, which shows that the left hand side does not depend on α. Now taking the limit $\alpha \to x$ through the rationals gives that $\varphi(x,y)$ is a function of xy, defined almost everywhere. This proof is expedited if φ is assumed to be continuous and the almost everywhere constraints may be dropped in favor of pointwise constraints. □

Next we define the family of n-Gaussians.

Definition 3.2. For $n \in \mathbb{N}$, the n-Gaussian is the function $g_n \in L^2(\mathbb{R})$ such that

$$g_n(x) = e^{-\frac{x^{2n}}{2n}}.$$

In analogy with the Fourier transform, we require in (1) that g_n be invariant under the integral operator Φ_n. The n-Gaussians behave as nearly ideal low-pass filters and as such are natural candidates for defining an integral transform. We denote the kernel of Φ_n by φ_n.

Our goal is to devise a family of operators that generalize the Fourier transform while retaining as many of its properties as possible. Specifically we look to an axiomatic characterization of the transforms. A similar axiomatic approach has been developed for the case of the Hilbert transform [6]. To this end, we inspect some properties of the Fourier kernel.

The most obvious properties of the Fourier kernel are that the real part is even, the imaginary part is odd and it is real analytic in both variables. There are multiple ways in which sine and cosine are related: as derivatives of each other, as distributional Hilbert transforms of each other and as linearly independent eigenfunctions of the Laplacian in one variable. There is no *a priori* obvious generalization of the derivative operator which enjoys many of the same properties so this is not feasible; the distributional Hilbert transform, while rich with theory, is difficult to treat in practice. For these reasons, and more, we choose to use view the relationship between sine and cosine as both being linearly independent solutions to the same differential equation.

We write φ_n as

$$\varphi_n(x, y) = c_n(x, y) + i s_n(x, y), \tag{3.2}$$

where c_n and s_n are real-valued. With these properties in mind, we state the following assumptions for φ_n:

(a) φ_n is of the form $\varphi_n(x, y) = f(xy)$ for some complex-valued function f,

(b) φ_n is real analytic,

(c) c_n is even and s_n is odd,

(d) c_n and s_n are linearly independent solutions to the same differential equation.

With the stipulated form for the integral kernel, (1) becomes

$$e^{-\frac{y^{2n}}{2n}} = \int_{-\infty}^{\infty} \varphi_n(xy) e^{-\frac{x^{2n}}{2n}}\, dx. \tag{3.3}$$

From this integral expression we can deduce some immediate consequences for φ_n. Without assuming evenness of c_n, it could not be uniquely determined from (3.3) as any odd, slowly-growing function can be added to it and the integration against the n-Gaussian would be unchanged. Additionally, s_n must be orthogonal to the n-Gaussians for all $y \in \mathbb{R}$, otherwise the right side of (3.3) would be complex whereas the left side is pure real. These observations support assumption (c) for φ_n. For now we will consider c_n as it can be easily established from (3.3).

Definition 3.3. Let $n \in \mathbb{N}$, $l \in \mathbb{N}_0$ and

$$c(n;l) = \frac{(-1)^l n}{(2n)^{2l+\frac{1}{2n}} \Gamma\left(l + \frac{1}{2n}\right) l!}.$$

We then define c_n as the entire function with the series

$$c_n(\eta) = \sum_{l=0}^{\infty} c(n;l) \eta^{2nl}\,, \qquad \eta \in \mathbb{C}. \tag{3.4}$$

Lemma 3.4. *Let the function c_n be as in (3.4), then, for $y \in \mathbb{R}$, it is real analytic and satisfies the integral equation*

$$e^{-\frac{y^{2n}}{2n}} = \int_{-\infty}^{\infty} c_n(xy) e^{-\frac{x^{2n}}{2n}}\, dx$$

Proof. Substituting the stipulated form for c_n into the integral equation gives

$$e^{-\frac{y^{2n}}{2n}} = \int_{-\infty}^{\infty} \sum_{l=0}^{\infty} c(n;l) y^{2nl} x^{2nl} e^{-\frac{x^{2n}}{2n}}\, dx. \tag{3.5}$$

If the series given by

$$\sum_{l=0}^{\infty} c(n;l) y^{2nl} \int_{-\infty}^{\infty} x^{2nl} e^{-\frac{x^{2n}}{2n}}\, dx \tag{3.6}$$

converges absolutely for all $y \in \mathbb{R}$, then the integral and summation in (3.5) can be interchanged by the Fubini-Tonelli theorem. Substituting $t = \frac{x^{2n}}{2n}$ in the integral in (3.6) yields

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2nl} e^{-\frac{x^{2n}}{2n}}\, dx &= \frac{1}{n}(2n)^{l+\frac{1}{2n}} \int_0^{\infty} t^{l+\frac{1}{2n}-1} e^{-t}\, dt \\ &= \frac{1}{n}(2n)^{l+\frac{1}{2n}} \Gamma\left(l + \frac{1}{2n}\right). \end{aligned}$$

Inserting this expression into (3.6) yields the following series

$$\sum_{l=0}^{\infty} \frac{1}{l!} \left(-\frac{y^{2n}}{2n}\right)^l. \tag{3.7}$$

This series converges absolutely for all $y \in \mathbb{R}$ and so the integration and summation can be interchanged in (3.5), resulting in $e^{-\frac{y^{2n}}{2n}}$ and thus the lemma is proved. □

In fact, for real η, c_n has the closed-form expression:

$$c_n(\eta) = \frac{1}{2}|\eta|^{n-\frac{1}{2}} J_{-1+\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right), \tag{3.8}$$

where J_ν is the Bessel function of the first kind of order ν [53, p. 40]. This can be checked directly by manipulating the Bessel function power series:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+\nu+1)m!} \left(\frac{z}{2}\right)^{2m+\nu}.$$

Since c_n can be expressed in terms of a Bessel function of the first kind, one may expect that c_n is the solution to a second-order differential equation. We prove this in the next proposition.

Proposition 3.5. *The function c_n as defined in* (3.4) *is a solution to the differential equation*

$$-\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}c_n(\eta) = c_n(\eta). \tag{3.9}$$

Proof. Since the series defined in (3.4) converges uniformly on compact sets, we can differentiate the series term-by-term. Hence we have that

$$\begin{aligned} -\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}c_n(\eta) &= \sum_{l=0}^{\infty} \frac{(-1)^l n}{(2n)^{2l+\frac{1}{2n}}\Gamma\left(l+\frac{1}{2n}\right) l!} \left(-\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}\right)\eta^{2nl} \\ &= \sum_{l=1}^{\infty} \frac{(-1)^l n}{(2n)^{2l+\frac{1}{2n}}\Gamma\left(l+\frac{1}{2n}\right) l!}(-2nl)(2nl-2n+1)\eta^{2nl-2n}. \end{aligned}$$

Upon reindexing the series and making use of the recursive property of the gamma function, this becomes $c_n(\eta)$ as claimed. □

In fact, a more general property holds. If $\mathscr{D}_n$ denotes the operator $-\frac{d}{dx}\frac{1}{x^{2n-2}}\frac{d}{dx}$, defined on sufficiently regular entire functions, then for fixed $y \in \mathbb{R}$, $\mathscr{D}_n(c_n(xy)) = y^{2n}c_n(xy)$. This equality can be checked in a similar manner as above.

Using the above result and the assumptions that s_n is real analytic and satisfies the same differential equation as c_n, we can now derive s_n—at least up to a multiplicative factor. We do so in the next proposition.

Proposition 3.6. *The function f defined by the series*

$$f(\eta) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2n)^{2l}\Gamma\left(l+2-\frac{1}{2n}\right) l!}\eta^{2nl+2n-1}$$

solves the differential equation $\mathscr{D}_n f(\eta) = f(\eta)$. Moreover, the solution set of entire functions to $\mathscr{D}_n g = g$ is spanned by c_n and f as defined above.

Proof. That $\mathscr{D}_n f = f$ follows via the same arguments in Proposition 2. Moreover, suppose that g is a solution to the differential equation $\mathscr{D}_n g = g$ and is given by the power series

$$g(\eta) = \sum_{l=0}^{\infty} \alpha_l \eta^l$$

for some $\alpha_l \in \mathbb{R}$. Then comparing the terms in the power series of $\mathscr{D}_n g$ and g, we get from

$$\begin{aligned}
-\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}g(\eta) &= -\frac{d}{d\eta}\frac{1}{\eta^{2n-2}}\frac{d}{d\eta}\sum_{l=0}^{\infty}\alpha_l\eta^l \\
&= -\sum_{l=1}^{2n-2}\alpha_l l(l-2n+1)x^{l-2n} - \sum_{l=2n}^{\infty}\alpha_l l(l-2n+1)x^{l-2n}
\end{aligned}$$

that $\alpha_1 = \cdots = \alpha_{2n-2} = \alpha_{2n+1} = \cdots = \alpha_{4n-2} = \cdots = 0$. Thus the only nonzero coefficients are α_{2nl} and α_{2nl-1} for some l. By solving the recursion relations for the coefficients, it follows that the solutions to $\mathscr{D}_n g = g$ are linear combinations of c_n and f since c_n and f are linearly independent solutions to the same second order differential equation. $\square$

With this result established, we make the following definition for s_n.

Definition 3.7. Let $n \in \mathbb{N}$, $l \in \mathbb{N}_0$ and

$$s(n;l) = -\frac{(-1)^l n}{(2n)^{2l+2-\frac{1}{2n}}\Gamma\left(l+2-\frac{1}{2n}\right) l!}.$$

We then define s_n as the entire function with the series

$$s_n(\eta) = \sum_{l=0}^{\infty} s(n;l)\eta^{2nl+2n-1}, \qquad \eta \in \mathbb{C}. \tag{3.10}$$

As noted above, s_n is only unique up to a multiplicative factor. The choice of s_n above guarantees unitarity; in fact, the only other choice that gives unitarity is $-s_n$, in exact agreement with Fourier transform theory.

Like c_n, s_n has a convenient representation in terms of a Bessel function of the first kind; particularly, we have that $s_n(\eta) = -\frac{1}{2}\operatorname{sgn}(\eta)|\eta|^{n-\frac{1}{2}} J_{1-\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right)$. If $n = 1$, $\varphi(\eta) = \frac{1}{\sqrt{2\pi}} e^{-i\eta}$ as expected.

With these representations in terms of Bessel functions, we can inspect the asymptotic behavior of φ_n easily. The Bessel function J_ν has the following asymptotic form [53, p. 199]:

$$J_\nu(\eta) \sim \sqrt{\frac{2}{\pi\eta}} \cos\left(\eta - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\eta^{-\frac{3}{2}}\right).$$

Hence c_n and s_n have the following asymptotic forms which will be useful in the analysis in the next section:

$$c_n(\eta) \sim \sqrt{\frac{n}{2\pi}} |\eta|^{\frac{n-1}{2}} \cos\left(\frac{|\eta|^n}{n} + \frac{\pi}{4}\left(1 - \frac{1}{n}\right)\right) + O\left(|\eta|^{-\frac{n+1}{2}}\right), \tag{3.11}$$

$$s_n(\eta) \sim \sqrt{\frac{n}{2\pi}} \operatorname{sgn}(\eta)|\eta|^{\frac{n-1}{2}} \cos\left(\frac{|\eta|^n}{n} - \frac{\pi}{4}\left(3 - \frac{1}{n}\right)\right) + \operatorname{sgn}(\eta) O\left(|\eta|^{-\frac{n+1}{2}}\right). \tag{3.12}$$

3.3 The Φ_n transform and its domain

When developing the Fourier transform in full generality, it is often first defined on functions in $L^1(\mathbb{R})$ and then extended by considering limits of Cauchy sequences in the dense subset $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ [43, 49] or $\mathcal{S}(\mathbb{R})$ of $L^2(\mathbb{R})$ [45]. For such functions, the results from the theory on $L^1(\mathbb{R})$ are true as well which streamlines many proofs. We employ a similar approach in the present setting with a caveat: because the kernels diverge at infinity, the function space on which the integral transforms are defined cannot be all of $L^1(\mathbb{R})$ but must be modified to mollify the growth of φ_n at infinity.

Define the measure $d\mu_n(x) = |x|^{\frac{n-1}{2}}\,dx$. We claim that for $f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$, $\int_{\mathbb{R}} |\varphi_n(xy)f(x)|\,dx$ is finite. In the case of $n = 1$, this space is identically $L^1(\mathbb{R})$ which is the usual space upon which the Fourier transform is defined. Let $y \in \mathbb{R}$ be fixed, $f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$ and $R \gg 1$, then

$$\begin{aligned}\int_{-\infty}^{\infty} |\varphi_n(xy)f(x)|\,dx &= \int_{|x|\le R} |\varphi_n(xy)||f(x)|\,dx + \int_{|x|>R} |\varphi_n(xy)||f(x)|\,dx \\ &\le M_1 \int_{|x|\le R} |f(x)|\,dx \\ &\quad + \sqrt{\frac{n}{2\pi}}|y|^{\frac{n-1}{2}} \int_{|x|>R} \left(|x|^{\frac{n-1}{2}} + O\left(|x|^{-\frac{n+1}{2}}\right)\right)|f(x)|\,dx.\end{aligned}$$

In the first term, we have used the fact that φ_n is continuous and hence bounded on compact sets. The first integral is then finite since $f \in L^1(\mathbb{R}, dx)$. In the second term, we have used the asymptotic form for φ_n as per (3.11) and (3.12). The integral of $|f|$ against $|x|^{\frac{n-1}{2}}$ in the second term is finite since $f \in L^1(\mathbb{R}, d\mu_n)$ by hypothesis. Moreover the integral of $|f|$ against $O(|x|^{-\frac{n+1}{2}})$ in the second term is finite since for some $M_2 > 0$, $O(|x|^{-\frac{n+1}{2}})|f(x)| \le M_2 R^{-\frac{n+1}{2}}|f(x)|$ and $f \in L^1(\mathbb{R}, dx)$. Thus for $f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$, $y \mapsto \int_{\mathbb{R}} \varphi_n(xy)f(x)\,dx$ is defined pointwise.

Since we are ultimately interested in an L^2 theory, it stands to reason that we should consider the space $L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n) \cap L^2(\mathbb{R}, dx)$. It is well-known that if $f \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$, then $\mathcal{F}f \in L^2(\mathbb{R}, dx)$; however this is not obviously true in general. Thus the natural function space upon which Φ_n acts, denoted $\operatorname{dom}\Phi_n$, is given by

$$\begin{aligned}\operatorname{dom}\Phi_n = \bigg\{ & f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n) \cap L^2(\mathbb{R}, dx) : \\ & y \mapsto \int_{\mathbb{R}} \varphi_n(xy) f(x)\, dx \in L^2(\mathbb{R}, dx) \bigg\}.\end{aligned} \tag{3.13}$$

This is clearly a vector space however we postpone discussion of its density in $L^2(\mathbb{R})$. With a formal domain, we may now define the Φ_n transform.

Definition 3.8. Let $f \in \operatorname{dom}\Phi_n$ and $y \in \mathbb{R}$, then $\Phi_n f$ is defined pointwise by

$$\Phi_n f(y) = \int_{-\infty}^{\infty} \varphi_n(xy) f(x)\, dx. \tag{3.14}$$

Clearly the dilation property (2) holds for $f \in \operatorname{dom}\Phi_n$ which a simple change of variable shows. Before showing analytic properties of Φ_n, we first explore some of its eigenfunctions as these will play an important role in the L^2 theory for Φ_n. We first explore the nature of a Riemann-Lebesgue lemma for Φ_n.

3.4 A Riemann-Lebesgue Lemma for Φ_n

Consider the function space $\mathfrak{X} = \{f \in C_c^\infty(\mathbb{R}) : 0 \notin \operatorname{supp} f\}$. Let $f \in \mathfrak{X}$, then $\mathscr{D}_n f$ will be well-defined since f and its derivatives are all zero in a neighborhood of 0. $\mathfrak{X}$ is dense in $L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$. Since f is compactly supported, $\Phi_n f$ is well-defined as an integral operator. Likewise, $\Phi_n \mathscr{D}_n f$ is well defined as an integral operator. We have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_n(xy) f(x)\, dx &= \frac{1}{y^{2n}} \int_{-\infty}^{\infty} \mathscr{D}_n \varphi_n(xy) f(x)\, dx \\ &= \frac{1}{y^{2n}} \int_{-\infty}^{\infty} \varphi_n(xy) \mathscr{D}_n f(x)\, dx.\end{aligned}$$

From the previous analysis, we know that $\int_{-\infty}^{\infty} \varphi_n(xy)\mathscr{D}_n f(x)\, dx$ grows as $|y|^{\frac{n-1}{2}}$ as a function of y. Combining this with the above expression shows that $\Phi_n f(y) \to 0$ as $y \to \pm\infty$.

Let $y \neq 0$ be a fixed real number and $f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$ and $g \in \mathfrak{X}$ be such that $\|f - g\| < \varepsilon y^{-\frac{n-1}{2}}$ in both the $L^1(\mathbb{R}, dx)$ and $L^1(\mathbb{R}, d\mu_n)$ norms for arbitrary $\varepsilon > 0$, then we have that

$$\begin{aligned}|\Phi_f(y)| &= \left| \int_{-\infty}^{\infty} \varphi_n(xy)(f(x) - g(x) + g(x))\, dx \right| \\ &\leq \int_{-\infty}^{\infty} |\varphi(xy)(f(x) - g(x))|\, dx + \int_{-\infty}^{\infty} |\varphi(xy) g(x)|\, dx.\end{aligned}$$

The second term goes to zero at infinity as noted above since $g \in \mathfrak{X}$. From the asymptotic form for φ_n, the first term can be bounded by

$$\begin{aligned}\int_{-\infty}^{\infty} \left(M + |xy|^{\frac{n-1}{2}}\right) |f(x) - g(x)|\, dx =& M \int_{-\infty}^{\infty} |f(x) - g(x)|\, dx \\ &+ |y|^{\frac{n-1}{2}} \int_{-\infty}^{\infty} |f(x) - g(x)||x|^{\frac{n-1}{2}}\, dx \\ \leq M\varepsilon |y|^{-\frac{n-1}{2}} + \varepsilon.\end{aligned}$$

Here M is a universal constant, independent of y. Letting y tend to infinity, we see that $\Phi_n f(y)$ tends to zero as all three terms go to zero.

3.5 Some eigenfunctions of Φ_n

We have already demonstrated one eigenfunction for Φ_n: g_n. From this, we can extract a family of eigenfunctions for Φ_n by implementing Akhiezer's technique since the kernel of Φ_n is of the form $\varphi_n(y,t) = f(xy)$. Since g_n is an eigenfunction of Φ_n by hypothesis,

$$e^{-\frac{y^{2n}}{2n}} = \int_{-\infty}^{\infty} \varphi_n(xy) e^{-\frac{x^{2n}}{2n}}\, dx.$$

Making the changes of variables $x = \alpha^{\frac{1}{2n}} r$ and $y = \alpha^{-\frac{1}{2n}} t$ where $\alpha > 0$, we see that φ_n is unchanged but we have

$$e^{-\frac{t^{2n}}{2n\alpha}} = \int_{-\infty}^{\infty} \varphi_n(rt) e^{-\alpha\frac{r^{2n}}{2n}} \alpha^{\frac{1}{2n}}\, dr.$$

Multiplying both sides by $\alpha^{-\frac{1}{4n}}$ yields the following

$$\alpha^{-\frac{1}{4n}} e^{-\frac{t^{2n}}{2n\alpha}} = \int_{-\infty}^{\infty} \varphi_n(rt) e^{-\alpha\frac{r^{2n}}{2n}} \alpha^{\frac{1}{4n}}\, dr.$$

We introduce the parameter $\beta = \frac{1}{\alpha}$ and note that $\alpha\frac{\partial}{\partial\alpha} = -\beta\frac{\partial}{\partial\beta}$. Thus

$$\left(-\beta\frac{\partial}{\partial\beta}\right)^m \left(\beta^{\frac{1}{4n}} e^{-\beta\frac{t^{2n}}{2n}}\right) = \int_{-\infty}^{\infty} \varphi_n(rt) \left(\alpha\frac{\partial}{\partial\alpha}\right)^m \left(\alpha^{\frac{1}{4n}} e^{-\alpha\frac{r^{2n}}{2n}}\right) dr. \qquad (3.15)$$

To eliminate the dependence upon the parameters α and β, after differentiating they may be set to 1. It is then clear that the even eigenfunctions are

$$\phi_{2m}^{(n)}(x) = \left(\alpha \frac{\partial}{\partial \alpha}\right)^m \left(\alpha^{\frac{1}{4n}} e^{-\alpha \frac{x^{2n}}{2n}}\right)\Bigg|_{\alpha=1}, \tag{3.16}$$

with eigenvalue $(-1)^m$. Particularly, Φ_n^2 acts as the identity on these functions.

Taking cues from the Fourier transform, the Hermite-Gauss functions, and noting that the lowest power in the series for $s_n(\eta)$ is η^{2n-1}, the obvious candidate for an odd eigenfunction of Φ_n is $x^{2n-1}e^{-\frac{x^{2n}}{2n}}$. To see that this is indeed an eigenfunction of Φ_n, note that

$$\int_{-\infty}^{\infty} \varphi_n(xy) x^{2n-1} e^{-\frac{x^{2n}}{2n}}\, dx = -i\, \mathrm{sgn}(y)|y|^{n-\frac{1}{2}} \int_0^{\infty} x^{3n-\frac{3}{2}} J_{1-\frac{1}{2n}}\left(\frac{|y|^n}{n} x^n\right) e^{-\frac{x^{2n}}{2n}}\, dx.$$

Letting $z = x^n$, this becomes

$$\int_{-\infty}^{\infty} \varphi_n(xy) x^{2n-1} e^{-\frac{x^{2n}}{2n}}\, dx = -\frac{i}{n}\, \mathrm{sgn}(y)|y|^{n-\frac{1}{2}} \int_0^{\infty} z^{2-\frac{1}{2n}} J_{1-\frac{1}{2n}}\left(\frac{|y|^n}{n} z\right) e^{-\frac{z^2}{2n}}\, dz.$$

This integral simplifies nicely [53, p. 394] to give

$$\int_{-\infty}^{\infty} \varphi_n(xy) x^{2n-1} e^{-\frac{x^{2n}}{2n}}\, dx = -iy^{2n-1} e^{-\frac{y^{2n}}{2n}}.$$

Hence $x^{2n-1}e^{-\frac{x^{2n}}{2n}}$ is an eigenfunction of Φ_n with eigenvalue $-i$. Repeating the same analysis as above with the even eigenfunctions, we obtain the following odd eigenfunctions with eigenvalue $(-1)^{m+1}i$:

$$\phi_{2m+1}^{(n)}(x) = x^{2n-1} \left(\alpha \frac{\partial}{\partial \alpha}\right)^m \left(\alpha^{1-\frac{1}{4n}} e^{-\alpha \frac{x^{2n}}{2n}}\right)\Bigg|_{\alpha=1}. \tag{3.17}$$

Unlike in the case of the even eigenfunctions, Φ_n^2 acts as the negative identity on the odd eigenfunctions.

Note that $\phi_m^{(n)} \in \operatorname{dom}\Phi_n$ for all m and n. Moreover $\phi_m^{(n)}$ has eigenvalue $(-i)^m$ under Φ_n. Since φ_n has polynomial growth and is continuous, $|\varphi_n(\eta)| \le M_1 + M_2|\eta|^\alpha$ for some $M_1, M_2, \alpha > 0$. Noting that $\phi_m^{(n)}$ has exponential decay, it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi_n(xy)\phi_m^{(n)}(x)\phi_{m'}^{(n)}(y)|\,dx\,dy \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(M_1 + M_2|xy|^\alpha)|\phi_m^{(n)}(x)\phi_{m'}^{(n)}(y)|\,dx\,dy < \infty.$$

Hence by Fubini-Tonelli, we have that

$$\langle \phi_m^{(n)}, \phi_{m'}^{(n)}\rangle = (-i)^m \langle \Phi_n\phi_m^{(n)}, \phi_{m'}^{(n)}\rangle = (-i)^m\langle \phi_m^{(n)}, \overline{\Phi_n\phi_{m'}^{(n)}}\rangle = (-i)^{m-m'}\langle \phi_m^{(n)}, \phi_{m'}^{(n)}\rangle,$$

and so if $m \not\equiv m' \pmod 4$, then $\langle \phi_m^{(n)}, \phi_{m'}^{(n)}\rangle = 0$. This is in direct analogy with the traditional Fourier transform eigenfunctions: there are four mutually orthogonal eigenspaces.

Furthermore, $\{\phi_m^{(n)}\}$ is a complete set of eigenfunctions. To see this, note that $\phi_m^{(n)}$ is of the form $p_m^{(n)}(x)e^{-\frac{x^{2n}}{2n}}$, where $p_m^{(n)}$ is a polynomial of degree $2mn$ or $2mn-1$; moreover, $p_m^{(n)}$ is a linear combination of powers of the form x^{2nl} or x^{2nl-1}, depending on whether m is even or odd.

We can employ Gram-Schmidt to obtain an orthonormal set from the eigenfunctions; the orthonormal set is denoted by $\widetilde{p}_m^{(n)}(x)e^{-\frac{x^{2n}}{2n}}$, where $\widetilde{p}_m^{(n)}$ is a polynomial of degree $2mn$ or $2mn-1$—in general, $p_m^{(n)}$ and $\widetilde{p}_m^{(n)}$ need not be the same. Additionally, the Gram-Schmidt procedure only occurs within each eigenspace since the different eigenspaces are mutually orthogonal by the preceding argument.

Because $p_{2k}^{(n)}$ is comprised of powers x^{2nl}, we can view $\widetilde{p}_{2k}^{(n)}(x)$ as a polynomial $\widetilde{q}_{2m}^{(n)}(x^{2n})$. The orthogonality of the functions $\widetilde{p}_{2k}^{(n)}(x)e^{-\frac{x^{2n}}{2n}}$ can then be summarized as

$$\int_{-\infty}^{\infty} \widetilde{p}_{2k}^{(n)}(x)\widetilde{p}_{2l}^{(n)}(x)e^{-\frac{x^{2n}}{n}}\,dx = 2n^{1-\frac{1}{2n}}\delta_{kl}.$$

After a change of variable, this becomes

$$\int_{0}^{\infty} x^{-1+\frac{1}{2n}}\widetilde{q}_{2k}^{(n)}(nt)\widetilde{q}_{2l}^{(n)}(nt)e^{-t}\,dx = \delta_{kl}.$$

The polynomials $\widetilde{q}_{2k}^{(n)}$ have degree k and so appropriate linear combinations show that every function of the form $x^k e^{-x}$ can be realized.

Proceeding in the same way for the odd eigenfunctions, we can view $\widetilde{p}_{2k+1}^{(n)}(x)$ as a polynomial $x^{2n-1}\widetilde{q}_{2k+1}^{(n)}(x^{2n})$. The orthogonality relation can again be summarized as

$$\int_{-\infty}^{\infty} \widetilde{p}_{2k+1}^{(n)}(x)\widetilde{p}_{2l+1}^{(n)}(x)e^{-\frac{x^{2n}}{n}}\,dx = \frac{1}{2}n^{-3+\frac{1}{n}}\delta_{kl}.$$

After making a change of variable, this becomes

$$\int_{0}^{\infty} x^{3-\frac{1}{n}}\widetilde{q}_{2k+1}^{(n)}(nt)\widetilde{q}_{2l+1}^{(n)}(nt)e^{-t}\,dx = \delta_{kl}.$$

Note that every monomial power appears like before.

Because $\widetilde{q}_k$ is a polynomial, the analysis by Akhiezer [2, p. 61] for the completeness of the Laguerre polynomials proves the completeness of eigenfunctions $\{\phi_m^{(n)}\}$ in $L^2(\mathbb{R})$. The completeness of the Laguerre polynomials can be summarized as follows:

Theorem 3.9. *Let $f : [0,\infty) \to \mathbb{R}$ be measurable and $\nu > -1$, then if*

1. $\displaystyle\int_0^\infty e^{-x}x^\nu |f(x)|^2\,dx < \infty$,

2. $\displaystyle\int_0^\infty e^{-x}x^\nu f(x)x^m\,dx = 0$

for all $m \in \mathbb{N}_0$, then $f \equiv 0$.

Furthermore, there is a convenient recursion relation for the $\phi_m^{(n)}$ which follows from (3.16) and (3.17):

$$\phi_{m+2}^{(n)}(x) = \frac{1}{4n}\phi_m^{(n)}(x) + \frac{x}{2n}\frac{d\phi_m^{(n)}}{dx}. \tag{3.18}$$

Because the eigenfunctions $\phi_m^{(n)}$ of Φ_n are complete and $\phi_m^{(n)} \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n) \cap L^2(\mathbb{R}, dx)$, it follows that $\operatorname{dom}\Phi_n$ is dense in $L^2(\mathbb{R})$.

3.6 Φ_n as an L^2 Isometry

We wish to show that Φ_n is an L^2 isometry on $\operatorname{dom}\Phi_n$. Traditionally, the L^2 isometry of the Fourier transform from $L^1(\mathbb{R}, dx)\cap L^2(\mathbb{R}, dx)$ to $L^2(\mathbb{R}, dx)$ is proved by appealing to the convolution theorem. However no obvious convolution theorem exists for Φ_n in general and so we take a purely L^2 approach by appealing to the completeness of the eigenfunctions of Φ_n.

Theorem 3.10. *If $f \in \operatorname{dom}\Phi_n$, $\|\Phi_n f\|_{L^2(\mathbb{R},dx)} = \|f\|_{L^2(\mathbb{R},dx)}$, so Φ_n is an isometry with dense range and extends to a unitary on $L^2(\mathbb{R}, dx)$.*

Proof. Let $\{\psi_m^{(n)}\}$ be an orthonormal basis of eigenfunctions of Φ_n. Such a basis exists by the analysis in Section 3.2. For $f \in \operatorname{dom}\Phi_n$, $\Phi_n f \in L^2(\mathbb{R}, dx)$ by hypothesis and so $\langle \Phi_n f, \psi_m^{(n)} \rangle$ is finite. Thus

$$\begin{aligned}\langle \Phi_n f, \psi_m^{(n)} \rangle &= \int_{-\infty}^{\infty} \Phi_n f(y) \overline{\psi_m^{(n)}(y)}\, dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi_n(xy) f(x)\, dx \right) \overline{\psi_m^{(n)}(y)}\, dy.\end{aligned}$$

We can interchange the integrals above since $y \mapsto \int_{\mathbb{R}} |\varphi_n(xy) f(x)|\, dx$ is finite everywhere and has at most polynomial growth at infinity and $\psi_m^{(n)}$ has exponential decay. Therefore

$$\langle \Phi_n f, \psi_m^{(n)} \rangle = \int_{-\infty}^{\infty} f(x) \overline{\int_{-\infty}^{\infty} \overline{\varphi_n(xy)} \psi_m^{(n)}(y)\, dy}\, dx.$$

It is clear that $\int_{\mathbb{R}} \overline{\varphi_n(xy)} \psi_m^{(n)}(y)\, dy = i^m \psi_m^{(n)}(x)$, giving the relation $\langle \Phi_n f, \psi_m^{(n)} \rangle = (-i)^m \langle f, \psi_m^{(n)} \rangle$. If we write $f = \sum_m \langle f, \psi_m^{(n)} \rangle \psi_m^{(n)}$, then

$$\begin{aligned}\Phi_n f &= \sum_m \langle \Phi_n f, \psi_m^{(n)} \rangle \psi_m^{(n)} \\ &= \sum_m (-i)^m \langle f, \psi_m^{(n)} \rangle \psi_m^{(n)}.\end{aligned}$$

The L^2 norm of $\Phi_n f$ gives $\|\Phi_n f\|^2_{L^2(\mathbb{R},dx)} = \sum_m |(-i)^m \langle f, \psi_m^{(n)} \rangle|^2 = \sum_m |\langle f, \psi_m^{(n)} \rangle|^2 = \|f\|^2_{L^2(\mathbb{R},dx)}$. Hence Φ_n is an L^2 isometry on $\operatorname{dom} \Phi_n$ which is dense in $L^2(\mathbb{R}, dx)$ and so Φ_n extends to an isometry on $L^2(\mathbb{R}, dx)$. Moreover, Φ_n has dense range in $L^2(\mathbb{R}, dx)$ since its range includes the span of the eigenfunctions $\{\phi_m^{(n)}\}$, thus Φ_n extends to a unitary on $L^2(\mathbb{R}, dx)$. $\square$

Remark 3.11. *This approach is, in some sense, a reverse of the usual approach for the Fourier transform. In traditional Fourier transform theory, the L^1 theory is* *explored, then the $L^1 \cap L^2$ theory, and culminates in the L^2 theory. Here, we briefly explore the L^1 theory, then establish some L^2 results, and ultimately make a remark about the $L^1 \cap L^2$ theory of Φ_n.*

In an abuse of notation, we denote the unitary extension of Φ_n to $L^2(\mathbb{R}, dx)$ by Φ_n though there is no risk of confusion as the meaning will be clear from context. Since the dilation property holds on $\operatorname{dom}\Phi_n$, Φ_n is bounded and $\mathcal{D}_\alpha$ is bounded, the dilation property holds for the unitary extension of Φ_n via simple continuity arguments.

3.7 The Spectrum of Φ_n

By analogy with the Fourier transform, we wish to show that Φ_n satisfies $\Phi_n^4 f = f$ for each $f \in L^2(\mathbb{R})$ which in turn gives that the spectrum of Φ_n is contained in $\{\pm 1, \pm i\}$.

Theorem 3.12. *$\Phi_n^4 = I$ on $L^2(\mathbb{R})$ and its spectrum is comprised only of $\pm 1, \pm i$.*

Proof. Let $f \in L^2(\mathbb{R})$ and $\{\psi_m^{(n)}\}$ be an orthonormal basis of eigenfunctions for Φ_n, then

$$\langle \Phi_n^4 f, \psi_m^{(n)} \rangle = \langle f, (\Phi_n^*)^4 \psi_m^{(n)} \rangle = \langle f, i^{4m} \psi_m^{(n)} \rangle = \langle f, \psi_m^{(n)} \rangle.$$

Since this holds for all m, it must be the case that $\Phi_n^4 f = f$, i.e. $\Phi_n^4 = I$. This gives that $\Phi_n^* = \Phi_n^{-1} = \Phi_n^3$ naturally. This generalizes the well-known result for the Fourier transform which states that $\mathcal{F}^* = \mathcal{F}^{-1} = \mathcal{F}^3$.

The spectral mapping theorem [45] shows that the spectrum of Φ_n is contained in $\{\pm 1, \pm i\}$. In fact, in Section 3 we demonstrated that each of these spectral values is realized and each is indeed an eigenvalue. $\square$

3.8 The Spectral Analysis of $\mathscr{D}_n$

The $\mathscr{D}_n$ operator is an unbounded operator as its spectrum is $[0,\infty)$ as evidenced by the fact that $\mathscr{D}_n(x)\varphi_n(xy) = y^{2n}\varphi_n(xy)$. The functions φ_n are generalized eigenfunctions [3], and so y^{2n} is in the continuous spectrum of $\mathscr{D}_n$. It is natural to inquire about the analytic properties of $\mathscr{D}_n$, namely whether or not it is (essentially) self-adjoint or positive.

Define $\mathscr{D}_n$ on the linear span of the eigenfunctions $\phi_m^{(n)}$. This space is dense by previous arguments and $\mathscr{D}_n$ is well-defined on these functions; particularly, it maps $\phi_m^{(n)}$ to another $L^2(\mathbb{R})$ function. A straightforward induction argument shows that $\mathscr{D}_n\phi_m^{(n)}$ is again a polynomial multiplying $\exp(-x^{2n}/2n)$. If $\phi_m^{(n)}(x) = p_m^{(n)}(x)\exp(-x^{2n}/2n)$, then $\mathscr{D}_n\phi_m^{(n)}$ has degree given by $\deg p_m^{(n)} + 2n$. For instance, $\exp(-x^{2n}/2n)$ is mapped to $(1-x^{2n})\exp(-x^{2n}/2n)$.

Suppose then that $\phi_m^{(n)}$ is not in the closure of the range of $\mathscr{D}_n + i$. Then for every $f \in \operatorname{span}\{\phi_{m'}^{(n)} : m' \in \mathbb{N}_0\}$

$$\langle(\mathscr{D}_n \pm i)f, \phi_m^{(n)}\rangle = 0. \tag{3.19}$$

Equivalently,

$$\langle f, (\mathscr{D}_n \mp i)\phi_m^{(n)}\rangle = 0 \tag{3.20}$$

for all $f \in \operatorname{span}\{\phi_{m'}^{(n)} : m' \in \mathbb{N}_0\}$. However $0 \neq f = (\mathscr{D}_n \mp i)\phi_m^{(n)} \in \operatorname{span}\{\phi_{m'}^{(n)} : m' \in \mathbb{N}_0\}$ by the above, so $\phi_m^{(n)}$ is in the closure of the range of $\mathscr{D}_n \pm i$. Thus the deficiency indices [37, p.138] for $\mathscr{D}_n$ are both zero and thus $\mathscr{D}_n$ is essentially self-adjoint. Hence the closure of $\mathscr{D}_n$ in the graph norm is a self-adjoint operator. Furthermore, $\overline{\mathscr{D}_n}$ is a positive operator as explained below.

By the spectral theorem, we know that there exists a unique unitary mapping

$\mathcal{U} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\mathcal{U}\overline{\mathscr{D}_n}g = \mathcal{M}_f\mathcal{U}g$ for some multiplication operator $\mathcal{M}_f$. By the analysis for Φ_n, we saw that the symbol for $\mathscr{D}_n$ under Φ_n was y^{2n}. Since $\mathscr{D}_n$ and $\overline{\mathscr{D}_n}$ agree on a dense set, we must conclude that the spectral family for $\overline{\mathscr{D}_n}$ is generated by Φ_n.

3.9 The Φ_n and Fourier-Bessel transforms

With the appearance of Bessel functions in the expression for φ_n, it is natural to ask what, if any, connection there is between Φ_n and the Fourier-Bessel transform. We choose to consider the following definition for the Fourier-Bessel transform:

$$\mathcal{F}_\nu f(y) = \int_0^\infty j_\nu(xy) f(x)\, d\lambda_\nu(x), \tag{3.21}$$

where $d\lambda_\nu(x) = x^{2\nu+1}\, dx$ and $j_\nu(x) = x^{-\nu} J_\nu(x)$. Most analysis of the Fourier-Bessel transform is restricted to the case $\nu > -\frac{1}{2}$ as in this range the measure $d\lambda_\nu$ is non-singular (c.f. [24]). Some analysis has been done in the regime $-1 < \nu < -\frac{1}{2}$, cf. [2, p. 62]. $\mathcal{F}_\nu$ is an isometry on $L^2(\mathbb{R}^+, d\lambda_\nu)$ when restricted to a dense subspace and also extends to a unitary on $L^2(\mathbb{R}, d\lambda_\nu)$.

Write $\Phi_n = \Phi_n^+ + i\Phi_n^-$, where Φ_n^+ is the integral operator with integral kernel c_n and Φ_n^- is the integral operator with integral kernel s_n. Φ_n^+ and Φ_n^- can be thought of as restrictions of Φ_n to even and odd functions, respectively. Thus Φ_n can be written as $\Phi_n = \Phi_n^+ \oplus i\Phi_n^-$, where we have decomposed $\operatorname{dom}\Phi_n$ into its even and odd subspaces.

To relate Φ_n to $\mathcal{F}_\nu$ we must project functions onto $\mathbb{R}^+$ since the Fourier-Bessel transform is restricted to $\mathbb{R}^+$. Let $\mathcal{P}^+$ denote the projection onto $\mathbb{R}^+$. If $f \in \operatorname{dom}\Phi_n$

is even, then there is a natural relationship between $\mathcal{P}^+ f$ and $\Phi_n f$: $\Phi_n f = \Phi_n^+ f = 2\Phi_n^+\mathcal{P}^+ f$. A similar relationship holds for odd functions. Thus we may restrict our attention to those $f \in \operatorname{dom}\Phi_n$ with support on $\mathbb{R}^+$ when considering Φ_n without loss of generality.

Define the operators $\mathcal{S}_n^+ : L^2(\mathbb{R}^+, dx) \to L^2(\mathbb{R}, d\lambda_{-1+\frac{1}{2n}})$ and $\mathcal{S}_n^- : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^+, d\lambda_{1-\frac{1}{2n}})$ by $\mathcal{S}_n^+ f(x) = n^{-\frac{1}{2}+\frac{1}{2n}} f(\sqrt[n]{nx})$ and $\mathcal{S}_n^- f(x) = n^{-\frac{1}{2}+\frac{1}{2n}} x^{-2+\frac{1}{n}} f(\sqrt[n]{nx})$. $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are both invertible and their inverses are given by a simple change of variable. Furthermore, $\Phi_n^+ = (\mathcal{S}_n^+)^{-1}\mathcal{F}_{-1+\frac{1}{2n}}\mathcal{S}_n^+$ and $\Phi_n^- = (\mathcal{S}_n^-)^{-1}\mathcal{F}_{1-\frac{1}{2n}}\mathcal{S}_n^-$. This gives the commutative diagrams shown in Figure 3.1.

It is straightforward to show that $\mathcal{S}_n^\pm$ are isometries so the fact that Φ_n is an isometry is a consequence of $\mathcal{F}_\nu$ being an isometry. Instead of simply using this fact from the outset, we chose to supply new proofs as the literature for $\mathcal{F}_\nu$ when $-1 < \nu < -\frac{1}{2}$ is quite sparce. While Φ_n is closely related to the Fourier-Bessel transform and many properties of Φ_n can be gleaned from the Fourier-Bessel transform, they are inherently different. Although there are extensions of the Fourier-Bessel transform to the whole real line (cf. [41]), there are no analogous generalizations of the Fourier-Bessel transform to the whole real line that are similar to Φ_n.

$$\begin{tikzcd}
L^2(\mathbb{R}^+, d\lambda) \arrow[r, "\Phi_n^+"] \arrow[d, "\mathcal{S}_n^+"'] & L^2(\mathbb{R}^+, d\lambda) \arrow[d, "\mathcal{S}_n^+"'] \\
L^2(\mathbb{R}^+, d\mu_{-1+\frac{1}{2n}}) \arrow[r, "\mathcal{F}_{-1+\frac{1}{2n}}"] & L^2(\mathbb{R}^+, d\mu_{-1+\frac{1}{2n}})
\end{tikzcd}$$

$$\begin{CD} L^2(\mathbb{R}^+, d\lambda) @>{\Phi_n^-}>> L^2(\mathbb{R}^+, d\lambda) \\ @V{\mathcal{S}_n^-}VV @VV{\mathcal{S}_n^-}V \\ L^2(\mathbb{R}^+, d\mu_{1-\frac{1}{2n}}) @>{\mathcal{F}_{1-\frac{1}{2n}}}>> L^2(\mathbb{R}^+, d\mu_{1-\frac{1}{2n}}) \end{CD}$$

Figure 3.1: Commutative diagrams showing the relationships between Φ_n^+ and Φ_n^- and the Fourier-Bessel transform.

3.10 The Short-Time Φ_n Transform

As a result of the linearity and exponential nature of the Fourier kernel, the Fourier transform of a translate of a function f differs from the Fourier transform of f by a modulation. There is unfortunately no similar relationship between the Φ_n transform of a function f and a translate of f. The lack of translation invariance is not a severe drawback as many integral transforms in practice do not have this, e.g. the Fourier-Bessel and Mellin transforms. Consequently, the most natural setting for the Φ_n transform is in fact as a short-time transform. Recall that the short-time Fourier transform (STFT) [26] of a function $f \in \mathcal{S}(\mathbb{R})$ with a window $g \in \mathcal{S}(\mathbb{R})$ is given by

$$\mathcal{V}_g f(x, y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ixy} g(x' - x) f(x')\, dx'. \tag{3.22}$$

Employing the notation $f_x(x') = f(x' - x)$, this can be rewritten in a more tangible form: $\mathcal{V}_g f(x, y) = \mathcal{F}(g_x f)(y)$. (3.22) can instead be written as $\mathcal{V}_g f(x, y) = e^{-ixy} \mathcal{F}(g f_{-x})(y)$, which can be interpreted as the Fourier kernel being centered with the window up to a phase factor. The second realization of the STFT will expedite the development of the short-time Φ_n transform.

Due to the translational invariance (up to a phase factor) of the Fourier transform, the window need not be centered with the kernel in the definition of the STFT since the power spectra for the two different formulations of the STFT given above are equivalent and thus carry the same information. However since the kernels for $n > 1$ are no longer translation invariant, some ambiguity arises when considering short-time analogues of Φ_n. We could consider two different definitions of the short-time Φ_n transform for a sufficiently nice windowing function g and function f:

$$\mathcal{V}_g^{(n)} f(x,y) = \int_{-\infty}^{\infty} \varphi_n(xy) g(x'-x) f(x')\, dx', \tag{3.23}$$

$$\mathcal{V}_g^{(n)} f(x,y) = \int_{-\infty}^{\infty} \varphi_n((x'-x)y) g(x'-x) f(x')\, dx'. \tag{3.24}$$

The former clearly resembles the STFT as given in (3.22), with φ and f centered at $x = 0$ and the window g, centered at x, passing over both. Despite their very different natures, the two notions are in fact equivalent up to an interchange of g and f and a reflection in the time-frequency plane. However the latter definition is more desirable than the former: the short-time Φ_n transforms of f and a translate of f as given by (3.24) differ only by a translation in the time-frequency plane; this is not true with the realization in (3.23).

Thus, we choose to break with the established literature of simply sliding the window across the kernel and function and instead choose to center the kernel with the window g and slide them across the function. That is, we choose the convention given in (3.24). We now give the formal definition of the short-time Φ_n transform and prove two theorems regarding the short-time Φ_n transform: the reconstruction property and an orthogonality relation.

Definition 3.13. Let $x, y \in \mathbb{R}$ and $g, f \in L^2(\mathbb{R})$ such that $gf_{-x} \in L^2(\mathbb{R})$ for all x.

We define the short-time Φ_n transform of f with window g to be

$$\mathcal{V}_g^{(n)} f(x, y) = \Phi_n(g f_{-x})(y). \tag{3.25}$$

If f and g are arbitrary functions in $L^2(\mathbb{R})$, $\Phi_n(gf_{-x})$ may not exist since gf_{-x} in general need not be in $L^2(\mathbb{R})$, thus the prescription that $gf_{-x} \in L^2(\mathbb{R})$ is necessary. This restriction is not very strong as it holds for all $f, g \in \mathcal{S}(\mathbb{R})$, which is a dense subspace of $L^2(\mathbb{R})$, but for the sake of mathematical rigor, we keep it. Assuming $\Phi_n(gf_{-x})$ exists in the original sense as an integral transform, e.g. if f and g are n-Gaussians, then the definition would be exactly as in (3.24). Instead of restricting to functions on which Φ_n is defined naturally as an integral transform and then extending the results via density arguments, we prefer to work in full generality from the outset for simplicity of argument. With this definition, we may immediately state the theorem.

Theorem 3.14. *Let $f, g \in L^2(\mathbb{R})$ such that $gf_{-x} \in L^2(\mathbb{R})$, then f may be reconstructed from $\mathcal{V}_g^{(n)} f$ by the following*

$$f(x) = \frac{1}{\langle g, g\rangle} \int_{-\infty}^{\infty} \overline{g(x - x')} \Phi_n \mathcal{V}_g^{(n)} f(x', -x + x')\, dx', \tag{3.26}$$

where $\Phi_n \mathcal{V}_g^{(n)} f$ is understood to be Φ_n acting on $h_{x'}(y) = \mathcal{V}_g^{(n)} f(x', y)$, i.e. x' is fixed.

Proof. We first consider the operation of Φ_n on $\mathcal{V}_g^{(n)} f$. This gives

$$\Phi_n \mathcal{V}_g^{(n)} f(x', -x + x') = \Phi_n(\Phi_n(g f_{-x'})(\cdot))(-x + x') = \Phi_n^2(g f_{-x'})(-x + x').$$

With the appearance of Φ_n^2, it is natural to break $gf_{-x'}$ into even and odd parts in order to make use of the fact that Φ_n^2 acts as the identity on even functions and the negative identity on odd functions. We write $f_{-x'} = f^+_{-x'} + f^-_{-x'}$ and $g = g^+ + g^-$. Therefore it follows that

$$\begin{aligned}\Phi_n \mathcal{V}_g^{(n)} f(x', -x + x') &= \Phi_n^2((g^+ + g^-)(f^+_{-x'} + f^-_{-x'}))(-x + x') \\ &= ((g^+ - g^-)(f^+_{-x'} - f^-_{-x'}))(-x + x') \\ &= g(x - x') f_{-x'}(x - x') \\ &= g(x - x') f(x).\end{aligned}$$

Then by the above,

$$\frac{1}{\langle g, g\rangle} \int_{-\infty}^{\infty} \overline{g(x - x')} \Phi_n \mathcal{V}_g^{(n)} f(x', -x + x')\, dx' = \frac{1}{\langle g, g\rangle} \int_{-\infty}^{\infty} \overline{g(x - x')} g(x - x') f(x)\, dx'.$$

This is readily realized as $f(x)$. Thus the theorem is proved. $\square$

The above theorem holds in more generality: one may instead replace Φ_n with any operator Φ such that Φ^2 acts as a parity operator.

With the ability to reconstruct a signal from its short-time Φ_n transform, it is natural to ask if energy is also preserved as is the case with the STFT. It so happens that an orthogonality relation holds regarding short-time Φ_n transforms—much like in the case of the STFT [26]—which immediately leads to energy preservation. We shall now state the theorem.

Theorem 3.15. *Let $f, \tilde{f}, g, \tilde{g} \in L^2(\mathbb{R})$ such that $g f_{-x}, \tilde{g}\tilde{f}_{-x} \in L^2(\mathbb{R})$, then the follow-*

ing orthogonality relation holds

$$\int_{\mathbb{R}^2} \mathcal{V}_g^{(n)} f(x,y)\overline{\mathcal{V}_{\tilde{g}}^{(n)} \tilde{f}(x,y)}\, dy\, dx = \langle f, \tilde{f}\rangle\langle g, \tilde{g}\rangle. \tag{3.27}$$

Proof. From the definition of the short-time Φ_n transform, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathcal{V}_g^{(n)} f(x,y)\overline{\mathcal{V}_{\tilde{g}}^{(n)} \tilde{f}(x,y)}\, dy\, dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_n(gf_{-x})(y)\overline{\Phi_n(\tilde{g}\tilde{f}_{-x})(y)}\, dy\, dx \\
&= \int_{-\infty}^{\infty} \langle \Phi_n(gf_{-x}), \Phi_n(\tilde{g}\tilde{f}_{-x})\rangle_y\, dx,
\end{aligned}$$

where the notation $\langle\cdot,\cdot\rangle_y$ is an inner product over y (with x fixed). Making use the unitarity of Φ_n, this becomes

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathcal{V}_g^{(n)} f(x,y)\overline{\mathcal{V}_{\tilde{g}}^{(n)} \tilde{f}(x,y)}\, dy\, dx &= \int_{-\infty}^{\infty} \langle gf_{-t}, \tilde{g}\tilde{f}_{-t}\rangle_y\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)\overline{\tilde{g}(y)} f_{-x}(y)\overline{\tilde{f}_{-x}(y)}\, dy\ dx \\
&= \int_{-\infty}^{\infty} g(y)\overline{\tilde{g}(y)} \int_{-\infty}^{\infty} f(x+y)\overline{\tilde{f}(x+y)}\, dx\, dy \\
&= \langle f, \tilde{f}\rangle\langle g, \tilde{g}\rangle.
\end{aligned}$$

Here we have employed Fubini's theorem. Taking $f = \tilde{f}$, $g = \tilde{g}$ and $\langle g, g\rangle = 1$, we see that $\|\mathcal{V}_g^{(n)} f\|^2_{L^2(\mathbb{R}^2,dx)} = \|f\|^2_{L^2(\mathbb{R},dx)}$ so the short-time Φ_n transform preserves energy. $\square$

The above theorem also holds in greater generality: one may replace Φ_n with any inner-product preserving operator Φ and the short-time transform arising from Φ will also be norm-preserving.

3.11 The Uncertainty Principle for Φ_n

With the eigenfunctions of Φ_n established, we may now investigate the nature of the uncertainty relation for Φ_n. Since x^{2n} plays the analogous role of x^2 in the theory of Φ_n, it stands to reason that the proper uncertainty product should be

$$\left(\int_{-\infty}^{\infty} x^{2n}|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} y^{2n}|\Phi_n f(y)|^2\,dy\right). \tag{3.28}$$

Since it is necessary to view Φ_n as an integral operator, we must have that $f \in \operatorname{dom}\Phi_n$, i.e. $f \in L^1(\mathbb{R},dx)\cap L^1(\mathbb{R},d\mu_n)\cap L^2(\mathbb{R},dx)$ such that $\Phi_n f \in L^2(\mathbb{R},dy)$. Moreover, we must have that $x \mapsto x^{2n}f(x) \in L^2(\mathbb{R},dx)$ and $y \mapsto y^{2n}\Phi_n f(y) \in L^2(\mathbb{R},dy)$ in order for the above inner products to be sensible.

In the case of the Fourier transform, the uncertainty principle is expedited by noting that if $f \in L^2(\mathbb{R},dx)$, then $f' \in L^2(\mathbb{R},dx)$ if and only if $y \mapsto iy\mathcal{F}f(y) \in L^2(\mathbb{R},dy)$, and moreover for y-a.e. $\mathcal{F}f'(y) = iy\mathcal{F}f(y)$. Since Φ_n was developed not from a first order differential operator but rather a second order differential operator, we instead appeal to the second derivative. By extending the logic above, $f'' \in L^2(\mathbb{R},dx)$ if and only if $y \mapsto y^2\mathcal{F}f(y) \in L^2(\mathbb{R},dy)$ and they are equal almost everywhere. This leads into the next lemma.

Lemma 3.16. *Let $f \in \operatorname{dom}\Phi_n \cap AC(\mathbb{R})$ and $x \mapsto \frac{1}{x^{n-1}}f'(x) \in L^2(\mathbb{R})$, then $\mathscr{D}_n f \in \operatorname{dom}\Phi_n$ if and only if $y \mapsto y^{2n}\Phi_n f(y) \in L^2(\mathbb{R},dy)$ and they are equal almost everywhere.*

Proof. Suppose that $\mathscr{D}_n f \in \operatorname{dom}\Phi_n$, then $\langle \Phi_n\mathscr{D}_n f, \psi_m^{(n)}\rangle$ is finite for all m, giving

$$\langle \Phi_n\mathscr{D}_n f, \phi_m^{(n)}\rangle = \langle \mathscr{D}_n f, \Phi_n^*\phi_m^{(n)}\rangle = \langle \mathscr{D}_n f, i^m\phi_m^{(n)}\rangle = \langle f, i^m\mathscr{D}_n\phi_m^{(n)}\rangle$$

The integration by parts with $\mathscr{D}_n$ is justified since $\phi_m^{(n)}$ has exponential decay and its derivative mitigates the singularity induced by $\mathscr{D}_n$. Applying Φ_n to both terms in the inner product and noting that $\Phi_n\mathscr{D}_n\phi_m^{(n)}(y) = y^{2n}\Phi_n\phi_m^{(n)}(y)$, we have that $\langle \Phi_n\mathscr{D}_n f, \phi_m^{(n)}\rangle = \langle y^{2n}\Phi_n f, \phi_m^{(n)}\rangle$. Since this inner product is finite for all $m \in \mathbb{N}_0$, it follows that $y \mapsto y^{2n}\Phi_n f(y) \in L^2(\mathbb{R})$ and moreover $\Phi_n\mathscr{D}_n f(y) = y^{2n}\Phi_n f(y)$ in the $L^2(\mathbb{R})$ sense. Thus they are equal almost everywhere. The converse follows by running the above argument in reverse. □

With the previous lemma, we are now able to formally state and prove the uncertainty principle for Φ_n.

Theorem 3.17. *Let $f \in \operatorname{dom}\Phi_n$ be L^2-normalized, $x \mapsto x^{2n}f(x) \in L^2(\mathbb{R})$ and $y \mapsto y^{2n}\Phi_n f(y) \in L^2(\mathbb{R})$, then*

$$\left(\int_{-\infty}^{\infty} x^{2n}|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} y^{2n}|\Phi_n f(y)|^2\,dy\right) \geq \frac{1}{4} \tag{3.29}$$

and equality holds if and only if $f(x) = e^{-\frac{\lambda x^{2n}}{2n}}$ for some $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$.

Proof. Since $y \mapsto y^{2n}\Phi_n f(y) \in L^2(\mathbb{R})$, Lemma 1 gives that $\mathscr{D}_n f \in L^2(\mathbb{R})$ and $y^{2n}\Phi_n f(y) = \Phi_n\mathscr{D}_n f(y)$ and so

$$\begin{aligned}\langle x^n f, x^n f\rangle\langle y^n\Phi_n f, y^n\Phi_n f\rangle &= \langle x^n f, x^n f\rangle\langle \Phi_n f, \Phi_n\mathscr{D}_n f\rangle \\ &= \langle x^n f, x^n f\rangle\langle f, \mathscr{D}_n f\rangle.\end{aligned}$$

Integrating by parts is justified by virtue of $f \in AC(\mathbb{R})$ and $\frac{1}{x^{n-1}}f'(x) \in L^2(\mathbb{R})$ and gives

$$\langle x^n f, x^n f\rangle\langle y^n\Phi_n f, y^n\Phi_n f\rangle = \langle x^n f, x^n f\rangle\langle x^{-n+1}f', x^{-n+1}f'\rangle. \tag{3.30}$$

Using Cauchy-Schwarz with $\eta(x) = x^n f(x)$ and $\xi(x) = x^{-n+1} f'(x)$, we have that

$$\langle x^n f, x^n f\rangle \langle y^n \Phi_n f, y^n \Phi_n f\rangle \geq |\langle xf, f'\rangle|^2, \tag{3.31}$$

with equality if and only if $\lambda x^n f(x) = \frac{1}{x^{n-1}} \frac{df}{dx}$, i.e. $f(x) = e^{-\frac{\lambda x^{2n}}{2n}}$ where $\Re(\lambda) > 0$. Note that the right hand side of (4.3) is bounded below by $\frac{1}{4}$ from the Fourier transform uncertainty principle. $\square$

3.12 A Family of Related Integral Transforms

The results in this section of the thesis are abridged as many of the results are carbon copies of the results in the previous section, with minor changes. Rather than repeating the arguments nearly exactly, the main points are summarized.

In the study of the Φ_n transforms, a second family of integral transforms arises, denoted by $\widetilde{\Phi}_n$. These transforms are very closely related to the Φ_n transforms. In the theory for the Φ_n transform, the integral kernel φ_n was defined via the differential equation that it solves, namely

$$-\frac{d}{dx}\frac{1}{x^{2n-2}}\frac{d}{dx}\varphi_n(x) = \varphi_n(x). \tag{3.32}$$

This relation may be written in another, more suggestive way (to be understood formally):

$$\left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)^* \left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)\varphi_n(x) = \varphi_n(x). \tag{3.33}$$

There is no *a priori* reason that one should investigate $\left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)^* \left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)$ rather than the operator $\left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)\left(\frac{1}{x^{n-1}}\frac{d}{dx}\right)^* = -\frac{1}{x^{n-1}}\frac{d^2}{dx^2}\frac{1}{x^{n-1}}$. The linearly independent, real analytic eigenfunctions of this operator, denoted $\widetilde{c}_n$ and $\widetilde{s}_n$, are given by

$$\widetilde{c}_n(\eta) = \frac{1}{2}\operatorname{sgn}(\eta)^{n-1}|\eta|^{n-\frac{1}{2}} J_{-\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right) \tag{3.34}$$

$$\widetilde{s}_n(\eta) = \frac{1}{2}\operatorname{sgn}(\eta)^{n}|\eta|^{n-\frac{1}{2}} J_{\frac{1}{2n}}\left(\frac{|\eta|^n}{n}\right). \tag{3.35}$$

A stark contrast between c_n and s_n and $\widetilde{c}_n$ and $\widetilde{s}_n$, respectively, is the appearance of the signum function in both $\widetilde{c}_n$ and $\widetilde{s}_n$. One may then define an integral kernel $\widetilde{\varphi}_n$ by

$$\widetilde{\varphi}_n(\eta) = \widetilde{c}_n(\eta) - i\widetilde{s}_n(\eta). \tag{3.36}$$

For $n = 1$, $\widetilde{\varphi}_n$ agrees with the Fourier kernel $\frac{1}{\sqrt{2\pi}}\exp(-i\eta)$. For $f \in L^1(\mathbb{R}, dx) \cap L^1(\mathbb{R}, d\mu_n)$, the $\widetilde{\Phi}_n$ transform of f is defined to be

$$\widetilde{\Phi}_n f(y) = \int_{-\infty}^{\infty} \widetilde{\varphi}_n(xy) f(x)\, dx. \tag{3.37}$$

There are two basic eigenfunctions for $\widetilde{\Phi}_n$: $x^{n-1}\exp(-x^{2n}/2n)$ and $2x^n\exp(-x^{2n}/2n)$. For $n = 1$ these agree exactly with the Gaussian and $x\exp(-x^2/2)$ which are known eigenfunctions of the Fourier transform. The kernel $\widetilde{\varphi}_n$ is a product kernel and so Akhiezer's technique for generating eigenfunctions may be applied. Doing so, we have the family of eigenfunctions

$$\widetilde{\phi}^{(n)}_{2m}(x) = \left(\frac{\partial}{\partial\alpha}\right)^m \left(\alpha^{\frac{1}{2}} x^{n-1} e^{-\frac{\alpha x^{2n}}{2n}}\right)\Bigg|_{\alpha=1} \tag{3.38}$$

$$\widetilde{\phi}^{(n)}_{2m+1}(x) = \left(\frac{\partial}{\partial\alpha}\right)^m \left(\alpha^{\frac{1}{2}+\frac{1}{2n}} x^{n} e^{-\frac{\alpha x^{2n}}{2n}}\right)\Bigg|_{\alpha=1}. \tag{3.39}$$

These are polynomials multiplying $\exp(-x^{2n}/2n)$, much like in the case of Φ_n with its eigenfunctions $\phi_m^{(n)}$. In fact, a minor modification of the previous argument shows that the eigenfunctions $\widetilde{\phi}_m^{(n)}$ have eigenvalue $\pm 1, \pm i$ under $\widetilde{\Phi}_n$ and are complete in $L^2(\mathbb{R})$. Thus $\widetilde{\Phi}_n$ extends to a unitary.

It is not surprising that there is a direct relationship between φ_n and $\widetilde{\varphi}_n$ when considering that the differential equations they satisfy are closely related. Direct computation shows that

$$\frac{1}{x^{n-1}} \frac{d}{dx} \varphi_n(x) = -i \widetilde{\varphi}_n(x) \tag{3.40}$$
$$\frac{d}{dx} \frac{1}{x^{n-1}} \widetilde{\varphi}_n(x) = -i \varphi_n(x). \tag{3.41}$$

As noted previously, a Fourier transform exits between a locally compact abelian group G and its dual group $\widehat{G}$. For $\chi \in \widehat{G}$ and $f \in L^1(G)$, the Fourier transform from G to $\widehat{G}$ is given by

$$\mathcal{F} f(\chi) = \int_G \overline{\chi(g)} f(g) \, dg.$$

For $g \in L^1(\widehat{G})$ and $g \in G$, the inverse Fourier transform from $\widehat{G}$ to G is given by

$$\mathcal{F}^{-1} f(g) = \int_{\widehat{G}} \chi(g) f(\chi) \, d\chi.$$

In the case of $(\mathbb{R}, +)$, this duality collapses to effectively a single structure as its dual group is again $(\mathbb{R}, +)$, leading to $\exp(-ixy)$ and $\exp(ixy)$ for the Fourier and inverse Fourier transform kernels, respectively. A similar pairing is seen in the setting pursued in this thesis: in the case of $n = 1$, Φ_n and $\widetilde{\Phi}_n$ reduce to the Fourier transform, but for $n \neq 1$, Φ_n and $\widetilde{\Phi}_n$ are different, yet closely related, integral operators.

Chapter 4

Supersymmetric Quantum Mechanics

4.1 The Quantum Mechanical Harmonic Oscillator

In the language of quantum mechanics, observables such as energy, momentum, and position are realized as densely-defined operators on a Hilbert space $\mathfrak{H}$ [17, p.26] which have specific representations in a given coordinate system. Energy is replaced with the Hamiltonian operator $\mathcal{H}$, momentum is replaced with the momentum operator p, and position is replaced with the position operator x. In spatial coordinates, the momentum operator is represented by $p = -i\hbar\nabla$, where ∇ is the gradient operator, and the position operator is represented by the multiplication operator $\mathcal{M}_x$, often simply denoted by x. Two important operations in quantum mechanics are the commutator and anticommutator. Given (possibly unbounded) densely-defined operators A, B on a Hilbert space $\mathfrak{H}$, their commutator and anticommutator are respectively given by

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA. \tag{4.1}$$

The existence of commutators is nontrivial in the case of unbounded operators [38, p.275]. In what follows, the existence of each is guaranteed however. In general, if both operators share an operator core, formal manipulation is justified. Unless stated otherwise or noted, this is assumed.

In this language, the energy equation for the classical harmonic oscillator in one variable becomes the operator

$$\mathcal{H}_{\mathrm{HO}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2,$$

where $\hbar$ is Planck's reduced constant, m is the mass of the oscillator, and k is the spring constant. A natural question to ask about any operator is what, if any, eigenvalues and eigenfunctions (or eigenvectors) it may have. To this end, we wish to solve the equation $\mathcal{H}_{\mathrm{HO}}\psi = \lambda\psi$. This results in the following second order linear differential equation after a suitable change of variables

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x) = \lambda\psi(x). \tag{4.2}$$

In the theory of differential equations, it is common to try series solution techniques for linear differential equations. In this one case, one may expand ψ as a series and find a recursion relation for its coefficients [50, p.251]. By requiring an L^2-normalizable solution, we get a constraint on the choices of λ and immediately determine the eigenvalues. This however is not a very enlightening technique and completely neglects the rich algebraic structure lurking behind the quantum-mechanical harmonic oscillator.

Perhaps the most elegant technique for solving this differential equation is via a factorization approach [17, p.136]. The operator $-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2$ resembles a difference of squares from elementary algebra and so it is natural to try to factor it with the same technique. The natural factorization might be

$$-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2 \stackrel{?}{=} \frac{1}{2}\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right). \tag{4.3}$$

However expanding the right hand side yields

$$\frac{1}{2}\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right) = -\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2+\frac{1}{2}\left(-\frac{d}{dx}x+x\frac{d}{dx}\right).$$

The term $-\frac{d}{dx}x+x\frac{d}{dx}$ can be readily simplified by noting that these are operators and must act on Hilbert space functions. Assuming $f \in L^2(\mathbb{R})$ and f' exists almost everywhere, then

$$-\frac{d}{dx}(xf)+x\frac{df}{dx} = -f(x)-xf'(x)+xf'(x) = -f.$$

Thus we may really view $-\frac{d}{dx}x+x\frac{d}{dx}$ as the operator $-I$ on such functions, and so our previous relation becomes

$$\frac{1}{2}\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right) = -\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2-\frac{1}{2}I. \tag{4.4}$$

The operators $\mathcal{H}_1 = -\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2$ and $\mathcal{H}_2 = \frac{1}{2}\left(-\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)$ will share many of the same properties, e.g. share eigenfunctions, and moreover their spectra will only be different by an overall shift of $\frac{1}{2}$ because,

$$\begin{aligned}
\sigma(\mathcal{H}_1) &= \{\lambda \in \mathbb{C} : \mathcal{H}_1 - \lambda I \text{ not invertible}\} \\
&= \left\{\lambda \in \mathbb{C} : \mathcal{H}_2 + \frac{1}{2}I - \lambda I \text{ not invertible}\right\} \\
&= \left\{\lambda \in \mathbb{C} : \mathcal{H}_2 - \left(\lambda - \frac{1}{2}\right) I \text{ not invertible}\right\} \\
&= \left\{\lambda + \frac{1}{2} \in \mathbb{C} : \mathcal{H}_2 - \lambda I \text{ not invertible}\right\} \\
&= \frac{1}{2} + \sigma(\mathcal{H}_2).
\end{aligned}$$

The factored Hamiltonian $\mathcal{H}_2$ is of a slightly simpler form, so we prefer to consider it from this point on. Many properties of $\mathcal{H}_2$ can be translated into properties of $\mathcal{H}_1$ without much difficulty as they only differ by an overall additive constant.

Define $\mathcal{H}$ then to be $\frac{1}{2}\left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)$ and define the operator a by

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right). \tag{4.5}$$

a is defined on those $f \in L^2(\mathbb{R})$ such that $f' \in L^2(\mathbb{R})$ and $xf \in L^2(\mathbb{R})$. If $f, g \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$ such that $f', g' \in L^2(\mathbb{R})$ and $xf, xg \in L^2(\mathbb{R})$, then the following equalities hold and are well-defined

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{d}{dx} + x\right) f(x)\overline{g(x)}\, dx &= \int_{-\infty}^{\infty} (f'(x) + xf(x))\overline{g(x)}\, dx \\
&= f(x)\overline{g(x)}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x)\overline{(-g'(x) + xg(x))}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\overline{\left(-\frac{d}{dx} + x\right) g(x)}\, dx.
\end{aligned}$$

Because $f\overline{g} \in AC(\mathbb{R}) \cap L^1(\mathbb{R})$, fg vanishes at infinity, justifying the final equality.

The operator $\frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)$ can be realized as the adjoint of a, and is denoted as a^*. Thus the Hamiltonian $\mathcal{H}$ may be realized as being equivalent to

$$\mathcal{H} = a^* a. \tag{4.6}$$

In our analysis, there was an implicit choice made when "factoring" the Hamiltonian $-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$. The factorization chosen was $\frac{1}{2}\left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)$, however there is no reason that an alternative factorization could not be made, particularly one may choose to instead factor the Hamiltonian as $\frac{1}{2}\left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right)$. Notice that $-\frac{d}{dx}$ now appears in the second term rather than the first. Expanding this product gives

$$\begin{aligned}
\frac{1}{2}\left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right) &= -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{1}{2}\left(\frac{d}{dx}x - x\frac{d}{dx}\right) \\
&= -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{1}{2}I.
\end{aligned}$$

Thus $\mathcal{H} = \frac{1}{2}\left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right) - I$ which is equivalent to $\mathcal{H} = aa^* - I$. Equating the two expressions for $\mathcal{H}$, it follows that

$$a^* a = aa^* - I. \tag{4.7}$$

In commutator language, this becomes $[a^*, a] = -I$.

This commutation relation achieves both a rich algebraic structure and a rich analytic structure. This commutation relation shows that the operators a, a^* and I generate a closed Lie algebra. This Lie algebra is the Heisenberg-Weyl Lie algebra [36, p.8]. Many algebraic structures underlying operators do not have associated analytic structures, cf. supersymmetry and supersymmetric quantum mechanics. This makes the harmonic oscillator unique as we will show that it also has a nice analytic structure.

Note first that the function $\psi_0(x) = \exp(-x^2/2) \in L^2(\mathbb{R})$ is annihilated by a, that is to say that $a\psi_0 = 0$:

$$\left(\frac{d}{dx} + x\right) e^{-\frac{x^2}{2}} = -xe^{-\frac{x^2}{2}} + xe^{-\frac{x^2}{2}} = 0. \tag{4.8}$$

Trivially then $a^*a\psi_0 = 0$ since $a\psi_0 = 0$. Thus a^*a has ψ_0 as an eigenfunction with eigenvalue 0. The function $\exp(x^2/2)$ is annihilated by a^* however it is not in $L^2(\mathbb{R})$, so while $aa^* \exp(x^2/2) = 0$ and so $a^*a \exp(x^2/2) = -\exp(x^2/2)$, it is not an eigenfunction as we are only interested in $L^2(\mathbb{R})$ eigenfunctions, not generalized eigenfunctions.

The commutation relation $[a^*, a] = -I$ yields two further commutation relations which will serve as the basis for the rest of the analysis of the quantum-mechanical harmonic oscillator. Applying a^* to a^*a on the left gives

$$a^*aa^* = a^*(a^*a + I) = a^*a^*a + a^*,$$

which may be realized as a commutation relation:

$$[a^*a, a^*] = a^*. \tag{4.9}$$

A similar approach shows that $[a^*a, a] = -a$. This leads into our next definition.

Definition 4.1. Let $0 \neq \lambda \in \mathbb{C}$ and $\mathcal{N}, \mathcal{X}$ be densely-defined operators on a Banach space $\mathfrak{X}$ be such that $[\mathcal{N}, \mathcal{X}] = \lambda\mathcal{X}$, $\operatorname{ran}\mathcal{X} \subseteq \operatorname{dom}\mathcal{N}$ and $\operatorname{ran}\mathcal{N} \subseteq \operatorname{dom}\mathcal{X}$. We say that $\mathcal{X}$ is a *ladder operator* for $\mathcal{N}$. If $\lambda \in \mathbb{R}$ and $\lambda > 0$, we say that $\mathcal{X}$ is a *raising operator* for $\mathcal{N}$. Likewise, if $\lambda \in \mathbb{R}$ and $\lambda < 0$, we say that $\mathcal{X}$ is a *lowering operator* for $\mathcal{N}$.

This is fundamentally a restatement of the notion of roots of Lie algebras cast in the language of unbounded operators [27, p.176]. Such commutation relations are important as they encode a lot of important information regarding the analytic structure of operators. This is discussed in the next theorem.

Theorem 4.2. *Let $0 \neq \lambda \in \mathbb{C}$ and $\mathcal{N}, \mathcal{X}$ be densely-defined operators on a Banach space $\mathfrak{X}$ be such that $[\mathcal{N}, \mathcal{X}] = \lambda\mathcal{X}$, $\operatorname{ran}\mathcal{X} \subseteq \operatorname{dom}\mathcal{N}$ and $\operatorname{ran}\mathcal{N} \subseteq \operatorname{dom}\mathcal{X}$. If ψ is an eigenvector for $\mathcal{N}$ with eigenvalue μ, then $\mathcal{X}\psi$ is also an eigenvector for $\mathcal{N}$ provided that $\mathcal{X}\psi \neq 0$.*

Proof. We have that

$$\mathcal{N}\mathcal{X}\psi = (\mathcal{X}\mathcal{N} + \lambda\mathcal{X})\psi = (\mu + \lambda)\mathcal{X}\psi. \tag{4.10}$$

Provided that $\mathcal{X}\psi \neq 0$, $\mathcal{X}\psi$ is also an eigenfunction of $\mathcal{N}$ with eigenvalue $\mu + \lambda$. $\square$

From the previous argument, a and a^* are ladder operators for a^*a, particularly a is a lowering operator and a^* is a raising operator for a^*a. This allows us to exactly solve for the eigenvalues for a^*a. Since 0 is an eigenvalue of a^*a, it follows that m is an eigenvalue for a^*a for any $m \in \mathbb{N}_0$.

Suppose then that λ is an eigenvalue of a^*a corresponding to eigenfunction ψ such that $m < \lambda < m + 1$ for some $m \in \mathbb{N}_0$. $a^m\psi$ is then an eigenfunction of a^*a with eigenvalue $0 < \lambda - m < 1$. Applying a once more yields another eigenfunction $a^{m+1}\psi$

which has eigenvalue $\lambda - m - 1 < 0$. However,

$$\begin{aligned}
0 &\leq \langle a^{m+2}\psi, a^{m+2}\psi\rangle \\
&= \langle a^* a(a^{m+1}\psi), a^{m+1}\psi\rangle \\
&= (\lambda - m - 1)\langle a^{m+1}\psi, a^{m+1}\psi\rangle \\
&< 0.
\end{aligned}$$

Thus, there cannot be an eigenfunction with eigenvalue λ such that $m < \lambda < m+1$. A not-too-dissimilar argument shows that the only eigenfunctions are exactly those of the form $(a^*)^m\psi_0$. We define then ψ_m by

$$\psi_m(x) = \frac{1}{2^{m/2}}\left(-\frac{d}{dx} + x\right)^m \psi_0(x). \tag{4.11}$$

A straightforward calculation shows that

$$\langle \psi_{m+1}, \psi_{m+1}\rangle = \sqrt{m}\langle \psi_m, \psi_m\rangle \tag{4.12}$$

so that $\psi_m \in L^2(\mathbb{R})$ for all m since $\psi_0 \in L^2(\mathbb{R})$.

It is a well-known fact that the entire spectrum of the quantum-mechanical harmonic oscillator is indeed comprised only of its eigenvalues [22, p.131], that is to say that there is no approximate point spectrum or residual spectrum. This is a direct consequence of the fact that the functions $\{\psi_m : m \in \mathbb{N}_0\}$ form a basis for $L^2(\mathbb{R})$.

4.2 Supersymmetric Quantum Mechanics

With the broad success of the factorization approach to the quantum-mechanical harmonic oscillator, physicists and mathematicians applied similar techniques to more general systems. In the quantum-mechanical harmonic oscillator, both a^*a and aa^* played a crucial role in the general theory. Let $\mathcal{H}_1$ be a one dimensional, time-independent Hamiltonian with potential V such that $\mathcal{H}_1$ is positive and has an eigenvalue of 0, then in the coordinate representation and neglecting physical constants, $\mathcal{H}_1$ may be written as

$$\mathcal{H}_1 = -\frac{1}{2}\frac{d^2}{dx^2} + V(x). \tag{4.13}$$

Provided that one is studying a system with finitely many negative energy states, the condition of positivity is not a significant assumption, as the operator $\mathcal{H}_1 + \lambda I$ will be positive for sufficiently large λ. Similarly, assuming that 0 is an eigenvalue is not a significant assumption by taking λ to be the negative of the lowest energy eigenvalue.

In the case of the quantum-mechanical harmonic oscillator, $V(x) = \frac{1}{2}x^2$. We define the operator $\mathcal{Q}$ and its (formal) adjoint $\mathcal{Q}^*$ on $L^2(\mathbb{R})$ by

$$\mathcal{Q} = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\right), \quad \mathcal{Q}^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right), \tag{4.14}$$

where it is further assumed that $\mathcal{H}_1 = \mathcal{Q}^*\mathcal{Q}$, that is to say that

$$V(x) = \frac{1}{2}W(x)^2 - \frac{1}{2}W'(x). \tag{4.15}$$

Here it is assumed that W is continuously differentiable. If W is also polynomially bounded from above and bounded from below, then the harmonic oscillator eigenfunctions are naturally in the domain of the operators $\mathcal{Q}$ and $\mathcal{Q}^*$ so that both are densely-defined. The image of the harmonic oscillator eigenfunctions will also be in the domain of $\mathcal{Q}^*$ and vice versa. Therefore if we restrict the domain of $\mathcal{Q}$ to the set of functions ψ such that $\mathcal{Q}\psi$ is in $L^2(\mathbb{R})$ and that $\mathcal{Q}^*\mathcal{Q}\psi$ is well-defined as a differential operator acting on ψ and in $L^2(\mathbb{R})$, our formal manipulations to follow are well-defined.

This is a Ricatti equation and is quite difficult to solve for general potentials V due to its nonlinear nature. If instead, the nodeless eigenfunction ψ_0 corresponding to eigenvalue 0 is known, we may define W by

$$W(x) = -\frac{d}{dx} \log \psi_0(x). \tag{4.16}$$

Direct computation shows that such a choice of W solves the Ricatti equation and that ψ_0 may be represented by

$$\psi_0(x) = A \exp\left(-\int_0^x W(x')\, dx' \right). \tag{4.17}$$

A simple application of Cauchy-Schwarz shows that if $\exp(-\int_0^x W(x')\, dx')$ is in $L^2(\mathbb{R})$, then $\exp(\int_0^x W(x')\, dx')$ cannot possibly be in $L^2(\mathbb{R})$. This is in tune with what was seen in the case of the quantum-mechanical harmonic oscillator.

Since aa^* was also critical in the analysis of the quantum-mechanical harmonic oscillator, it is natural to inspect the nature of $\mathcal{Q}\mathcal{Q}^*$ as well. The essence of supersymmetric quantum mechanics is the analysis of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ and the relations between them. Defining $\mathcal{H}_2 = \mathcal{Q}\mathcal{Q}^*$ yields the operator

$$\mathcal{H}_2 = \mathcal{Q}\mathcal{Q}^* = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}W(x)^2 + \frac{1}{2}W'(x). \tag{4.18}$$

Note that we need not duplicate the argument above for $\mathcal{Q}^*$ since $\mathcal{Q}$ and $\mathcal{Q}^*$ are of effectively the same form and so if $\mathcal{Q}^*\mathcal{Q}\psi$ is well-defined, so will be $\mathcal{Q}\mathcal{Q}^*\psi$. Note that both operators are densely-defined.

In the quantum-mechanical harmonic oscillator, $[a^*, a] = -I$; assuming a form of $a = \frac{d}{dx} + W(x)$, direct computation shows that $[a^*, a] = -I$ if and only if $W(x) = x + \alpha$ for some $\alpha \in \mathbb{R}$. Thus, in general, supersymmetric quantum mechanics should not be expected to have as rich an analytic structure as the quantum-mechanical harmonic oscillator. Indeed, many quantum-mechanical systems do not have nice algebraic relations between their eigenvalues, and so a true ladder structure will not exist involving the operators $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{Q}$, and $\mathcal{Q}^*$.

Instead, a different structure exists for supersymmetric quantum mechanics. It is straightforward that $\mathcal{Q}\mathcal{H}_1 = \mathcal{H}_2\mathcal{Q}$ and $\mathcal{H}_1\mathcal{Q}^* = \mathcal{Q}^*\mathcal{H}_2$. These intertwining relations lead to the well-known intertwining between the eigenvalues and eigenfunctions of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Theorem 4.3. *Let ψ be an eigenfunction of $\mathcal{H}_1$ with eigenvalue $\lambda \neq 0$, then $\mathcal{Q}\psi$ is an eigenfunction of $\mathcal{H}_2$ with eigenfunction λ.*

Proof. First note that $\langle \mathcal{Q}\psi, \mathcal{Q}\psi \rangle = \langle \mathcal{Q}^*\mathcal{Q}\psi, \psi \rangle = \lambda \langle \psi, \psi \rangle < \infty$ so $Q\psi$ is a permissible eigenfunction of $\mathcal{H}_2$ since it is in $L^2(\mathbb{R})$ and $\mathcal{Q}\psi \neq 0$. Moreover, $\mathcal{H}_2\mathcal{Q}\psi = \mathcal{Q}\mathcal{H}_1\psi = \lambda\mathcal{Q}\psi$. Hence $\mathcal{Q}\psi$ is an eigenfunction of $\mathcal{H}_2$ with eigenvalue of λ. The assumption that $\lambda \neq 0$ is critical as it prevents $\mathcal{Q}\psi = 0$. □

A similar argument shows that if φ is an eigenfunction of $\mathcal{H}_2$ with eigenvalue λ, then $\mathcal{Q}\varphi$ is an eigenfunction of $\mathcal{H}_1$ with eigenvalue λ. The case of $\mathcal{Q}^*\psi = 0$ is of no consequence since if $\exp\left(-\int_0^x W(x')\,dx'\right)$ is in $L^2(\mathbb{R})$, $\exp\left(\int_0^x W(x')\,dx'\right)$ cannot be in $L^2(\mathbb{R})$.

The supersymmetric operators $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{Q}$, and $\mathcal{Q}^*$ can be combined into one framework via a matricial representation on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Define matrix operators $\mathbb{H}$, $\mathbb{Q}$ and $\mathbb{Q}^*$ by

$$\mathbb{H} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix}, \quad \mathbb{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{Q} & 0 \end{pmatrix}, \quad \mathbb{Q}^* = \begin{pmatrix} 0 & \mathcal{Q}^* \\ 0 & 0 \end{pmatrix}. \tag{4.19}$$

These operators enjoy an algebraic structure:

$$[\mathbb{H}, \mathbb{Q}] = 0, \qquad [\mathbb{H}, \mathbb{Q}^*] = 0 \tag{4.20}$$

$$\{\mathbb{Q}, \mathbb{Q}^*\} = \mathbb{H}, \qquad \{\mathbb{Q}, \mathbb{Q}\} = 0 = \{\mathbb{Q}^*, \mathbb{Q}^*\}. \tag{4.21}$$

This is a $\mathbb{Z}_2$-graded Lie superalgebra [34, 15] where the grading is generated by the subspaces $\mathfrak{X}_1 = \operatorname{span}\mathbb{H}$ and $\mathfrak{X}_2 = \operatorname{span}\{\mathbb{Q}, \mathbb{Q}^*\}$.

Diagrammatically, the intertwining for $\mathcal{H}_1$ and $\mathcal{H}_2$ can be easily summarized. Let $\{\psi_m : m \in \mathbb{N}_0\}$ and $\{\varphi_m : m \in \mathbb{N}_0\}$ denote the eigenfunctions for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In the language of supersymmetry, the eigenstates of $\mathcal{H}_1$ and $\mathcal{H}_2$ form what are called *sectors*. Figure 4.1 shows the structure relating the two sectors.

As seen above, for supersymmetric quantum mechanics, the "ladder" structure is quite different from the quantum-mechanical harmonic oscillator insofar that the charge operators only allow one to move left and right between sectors, rather than

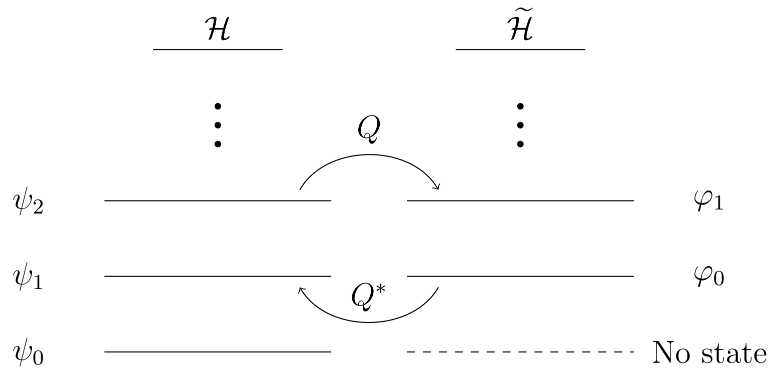

Figure 4.1: The ladder structure for supersymmetric quantum mechanics.

up and down within either sector. In supersymmetric quantum mechanics, one often defines a hierarchy of Hamiltonians by repeating the above process with φ_0 as the new ground state. This creates a series of Hamiltonians which are intertwined via the charge operators, however again one may only move left and right between sectors, thus it is not a true ladder structure. In the next chapter, a new structure is proposed to remedy this which melds the graded structure of supersymmetry (and supersymmetric quantum mechanics) with the analytic structure of the harmonic oscillator. Particularly, knowing the ground states for each sector, one may generate all higher excited states.

Chapter 5

Coupled Supersymmetric Quantum Mechanics and Associated Integral Operators

In Chapter 3 generalized Fourier transforms that emerged from an axiomatic framework were investigated. These were identified as the unitary transforms that diagonalized certain singular Laplacian operators $\Delta_n = -\frac{d}{dx}\frac{1}{x^{2n-2}}\frac{d}{dx}$. The motivation for the present work came from the question whether these transforms are related to a type of oscillator Hamiltonian—or Hamiltonian-like operator. This is indeed the case. If $a_n = \frac{1}{\sqrt{2}}\left(\frac{1}{x^{n-1}}\frac{d}{dx} + x^n\right)$, then $\mathcal{H} = a_n^* a_n$ has the set of eigenvalues $\{2kn, 2kn + 2n - 1\}_{k=0}^{\infty}$ and the corresponding eigenfunctions are complete in $L^2(\mathbb{R})$.

While supersymmetric quantum mechanics has a somewhat nice *algebraic* structure, the algebraic structure is not strong enough to give meaningful information about the analytic structure of the underlying Hamiltonians. In this chapter, we develop a family of supersymmetric systems which have a rich analytic structure and also a richer algebraic structure than traditional supersymmetric quantum mechanics. We also develop analogues of the Fourier transform for these systems and present further generalizations thereof.

5.1 Coupled Supersymmetry and Its Lie Algebra

In Chapter 3, it was shown that the QMHO has two separate factorizations: $\mathcal{H}_{\mathrm{HO}} = a^*a + \frac{1}{2}$ and $\mathcal{H}_{\mathrm{HO}} = aa^* - \frac{1}{2}$. Defining then the SUSY Hamiltonian $\mathcal{H}_1 = a^*a$, its SUSY partner Hamiltonian is $\mathcal{H}_2 = aa^*$. From the two factorizations of the QMHO, it is clear that $\mathcal{H}_1$ and $\mathcal{H}_2$ each have two distinct factorizations, even though these two factorizations are intimately connected.

That $\mathcal{H}_2 = \mathcal{H}_1 + 1$ is a restatement of the commutation relation for a and a^*—which is equivalent to the canonical commutation relation. The canonical commutation relation can be a very rigid structure [37, p.274] and as such cannot be a point of generalization. Note that the canonical commutation relations do not necessarily guarantee that $[a^*, a] = -I$ is solved by $a = x + ip$, see [37, p.275]. The Schrödinger representation is not strong enough to guarantee uniqueness, however the exponentiated (group) representation is. As the canonical commutation relations can be very rigid, we instead use the property that the QMHO and its partner Hamiltonian both have two distinct factorizations to develop our theory and this leads into our first definition.

Definition 5.1. Let a, b be closed, densely defined operators on a Hilbert space $\mathfrak{H}$, a^* and b^* be their adjoints, and $\gamma, \delta \in \mathbb{R}$ with $\gamma < \delta$. Furthermore, suppose that $\operatorname{dom} a = \operatorname{dom} b$, $\operatorname{dom} a^* = \operatorname{dom} b^*$, $\operatorname{ran} a \subseteq \operatorname{dom} a^*$ (and vice versa), and similarly for b and b^*. The ordered quadruplet $\{a, b, \gamma, \delta\}$ defines a *coupled supersymmetric system* (*coupled SUSY system*) if it satisfies

$$a^*a = b^*b + \gamma, \tag{5.1}$$

$$aa^* = bb^* + \delta. \tag{5.2}$$

The operators a^*a, aa^*, b^*b and bb^* will be referred to as Hamiltonians throughout the chapter.

It is easily seen that the coupled SUSY conditions are equivalent to the system of equations for commutators $[a^*, a] = [b^*, b] + \gamma - \delta$ and anticommutators $\{a^*, a\} = \{b^*, b\} + \gamma + \delta$. Note that (5.1) and (5.2) do not imply each other, as evidenced by taking $a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)$ and $b = \frac{1}{\sqrt{2}}e^{ix}\left(\frac{d}{dx} + x\right)$. In this case, (5.1) holds whereas (5.2) does not. Placing the exponential on the right in the definition of b shows that the reverse implication does not hold either.

The condition that $\gamma \neq \delta$ will play a crucial role in what follows. We assumed without loss of generality that $\gamma < \delta$; otherwise, if $\gamma > \delta$, we may simply switch the roles of a and a^* as well as b and b^* and the conditions for Definition 1 would still hold.

Example 5.2. Definition 1 includes the QMHO by letting $b^* = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) = a$, $-\gamma = 1 = \delta$. There exists an infinite family of examples satisfying the coupled SUSY equations. A straightforward calculation shows that, for $n \in \mathbb{N}$, the operators

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{1}{x^{n-1}}\frac{d}{dx} + x^n\right) \tag{5.3}$$

$$b_n = \frac{1}{\sqrt{2}}\left(-\frac{1}{x^{n-1}}\frac{d}{dx} + x^n\right) \tag{5.4}$$

taken with their adjoints on $L^2(\mathbb{R})$ also define a coupled SUSY system when restricted to an appropriate subspace where $\gamma = -1$ and $\delta = 2n-1$. This family of examples is closely related to standard SUSY with the anharmonic superpotential $W_1(x) = x^{2n-1}$ where the charge operator $\frac{d}{dx} + x^{2n-1}$ has been multiplied on the left by $\frac{1}{x^{n-1}}$. This relationship has connections to canonical transformations and mutually unbiased bases [31].

In SUSY, one considers broken and unbroken (or exact) systems. SUSY is unbroken if at least one of $\mathcal{Q}_1$ and $\mathcal{Q}_1^*$ in the factorization $\mathcal{H}_1 = \mathcal{Q}_1^*\mathcal{Q}_1$ annihilates a state and is broken if neither annihilate a state [14]. Unbroken SUSY is the primary focus in the study of basic supersymmetric quantum mechanics and is the focus of much of the remainder of this chapter. To this end, we make the following definition.

Definition 5.3. Let $\{a, b, \gamma, \delta\}$ form a coupled SUSY system. We say that it is an *unbroken* coupled SUSY system if a annihilates a state and b^* annihilates a state. Otherwise we say that it is a *broken* coupled SUSY system.

Remark 5.4. *If $\{a, b, \gamma, \delta\}$ form an unbroken coupled SUSY system, it cannot be the case that a^* and b annihilate states since otherwise $\delta < \gamma$, contradicting the assumption that $\gamma < \delta$. If however* (5.1) *and* (5.2) *hold but a^* and b annihilate states, we may reverse the roles of a and a^* (and b and b^*) to obtain an unbroken coupled SUSY system.*

Coupled SUSY, as the name suggests, automatically comes with a SUSY structure by noting that $\mathcal{H}_1 = a^*a$ and $\mathcal{H}_2 = aa^*$ come equipped with the usual SUSY ladder structure. In our first theorem, we show that a ladder structure exists for coupled SUSY.

Theorem 5.5. *If* $\{a, b, \gamma, \delta\}$ *form a coupled SUSY system, then* a^*b *and* b^*a *act as ladder operators for* a^*a *(and* b^*b*) while* ab^* *and* ba^* *act as ladder operators for* aa^* *(and* bb^**). Moreover, the triples* $\{a^*a - \frac{\gamma}{2}, a^*b, b^*a\}$ *and* $\{aa^* - \frac{\delta}{2}, ab^*, ba^*\}$ *generate Lie algebras isomorphic to* $\mathfrak{su}(1,1)$*.*

Proof. To prove this, we proceed in much the same way as in the standard QMHO by considering the commutator of a^*a with a^*b, a^*a with b^*a, and a^*b with b^*a. The other cases with aa^*, ab^* and ba^* follow the same logic and so they are omitted for the sake of brevity.

$$\begin{aligned}
[a^*a, a^*b] &= a^*aa^*b - a^*ba^*a \\
&= a^*aa^*b - a^*b(b^*b + \gamma) \\
&= a^*(aa^* - bb^*)b - \gamma a^*b \\
&= (\delta - \gamma)a^*b.
\end{aligned}$$

Similar reasoning shows that $[a^*a, b^*a] = -(\delta - \gamma)b^*a$. Since $\gamma < \delta$, a^*b is a raising operator for a^*a and b^*a is a lowering operator for a^*a. To show that these generate a Lie algebra, we inspect the commutator of a^*b and b^*a:

$$\begin{aligned}
[a^*b, b^*a] &= a^*bb^*a - b^*aa^*b \\
&= a^*(aa^* - \delta)a - b^*(bb^* + \delta)b \\
&= (a^*a)^2 - (b^*b)^2 - \delta(a^*a + b^*b) \\
&= 2(\gamma - \delta)\left(a^*a - \frac{\gamma}{2}\right).
\end{aligned}$$

Thus, the triple generates a Lie algebra as it is closed under commutation. After adding a multiple of the identity to a^*a and rescaling,

$$\mathcal{K}_+ = \frac{1}{\delta-\gamma}a^*b, \quad \mathcal{K}_- = \frac{1}{\delta-\gamma}b^*a, \quad \mathcal{K}_0 = \frac{1}{\delta-\gamma}\left(a^*a - \frac{\gamma}{2}\right) \tag{5.5}$$

are seen to verify

$$[\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \quad [\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0 \tag{5.6}$$

the commutation relations of $\mathfrak{su}(1,1)$ [36], hence the Lie algebra associated to coupled SUSY is found to be isomorphic to the $\mathfrak{su}(1,1)$ Lie algebra. $\square$

We note that the discrete series of $\mathfrak{su}(1,1)$ representations correspond to a choice of $\frac{\gamma}{\delta-\gamma} \in \mathbb{Z}$, $\frac{\gamma}{\delta-\gamma} \leq -2$ [36].

As with any mathematical structure, it is of interest to ask is if there are systems for which the commutation relations in Theorem 1 hold that do not correspond to a coupled SUSY system—be it a broken or unbroken coupled SUSY. In the next theorem, we prove that this is not the case under the modest assumption that $\ker a^* = \{0\} = \ker b$.

Theorem 5.6. *If a, a^*, b, b^* are operators satisfying $\ker a^* = \{0\} = \ker b$ and*

$$[a^*a, a^*b] = \lambda a^*b, \tag{5.7}$$

$$[bb^*, ba^*] = \lambda' ba^*, \tag{5.8}$$

$$[a^*b, b^*a] = \mu a^*a + \nu, \tag{5.9}$$

$$[ba^*, ab^*] = \mu' aa^* + \nu', \tag{5.10}$$

where $\lambda, \lambda', \nu, \nu' \in \mathbb{R}$, $\lambda \neq 0$, then $b^*b = \alpha a^*a + \gamma$ and $bb^* = \beta aa^* + \delta$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Proof. Suppose that $a^*a = b^*b + S$ and $aa^* = bb^* + T$ for some as-of-yet undetermined operators S and T. Inspecting commutation relations, we have

$$\begin{aligned}
[a^*a, a^*b] &= a^*(aa^*b - ba^*a) \\
&= a^*((bb^* + T)b - b(b^*b + S)) \\
&= a^*(Tb - bS).
\end{aligned}$$

Likewise, it follows that

$$[bb^*, ba^*] = b(a^*T - Sa^*).$$

Equating the above with the ladder operator relations, it can be seen that

$$\begin{aligned}
[a^*a, a^*b] &= \lambda a^*b = a^*(Tb - bS) \\
[bb^*, ba^*] &= \lambda' ba^* = b(a^*T - Sa^*).
\end{aligned}$$

Since a^* and b have trivial kernel, we must have that $Tb - bS = \lambda b$ and similarly $a^*T - Sa^* = \lambda' a^*$. We may use these to prove our result.

$$\begin{aligned}
[ba^*, ab^*] &= b(b^*b + S)b^* - a(a^*a - S)a^* \\
&= (bb^*)^2 + bSb^* - (aa^*)^2 + aSa^* \\
&= (aa^* - T)^2 + bSb^* - (aa^*)^2 + aSa^* \\
&= -aa^*T - Taa^* + T^2 + bSb^* + aSa^* \\
&= -aa^*T - T(bb^* + T) + T^2 + bSb^* + aSa^* \\
&= -a(a^*T - Sa^*) - (Tb - bS)b^* \\
&= -\lambda' aa^* - \lambda bb^*
\end{aligned}$$

Making use of our relations above, it follows that

$$[ba^*, ab^*] = \mu' aa^* + \nu' = -\lambda' aa^* - \lambda bb^*.$$

Therefore

$$bb^* = -\frac{\lambda' + \mu'}{\lambda} aa^* - \frac{\nu'}{\lambda} \tag{5.11}$$

By repeating these steps with $[a^*b, b^*a]$ and using the above relations $Tb - bS = \lambda b$ and $a^*T - Sa^* = \lambda' a^*$, $[a^*b, b^*a] = -\lambda b^*b - \lambda' a^*a = \mu a^*a + \nu$ the claimed affine relationship between a^*a and b^*b holds as well. □

Taking $\lambda = \lambda'$ in the previous theorem gives exactly that a, b form a coupled SUSY system as $\alpha = 1 = \beta$ in this case.

It is natural to ask if a coupled SUSY system is unique, i.e., given a, γ, δ if b is unique or if there are many possible choices for b. We prove a uniqueness result in the next theorem.

Theorem 5.7. *Suppose that $\{a, b, \gamma, \delta\}$ and $\{a, c, \gamma, \delta\}$ define coupled SUSY systems. Then $b = U(c^*c)^{1/2}$ and $c = V(b^*b)^{1/2}$ for some partial isometries U, V.*

Proof. Since we have that

$$b^*b + \gamma = a^*a = c^*c + \gamma, \tag{5.12}$$

$$bb^* + \delta = aa^* = cc^* + \delta, \tag{5.13}$$

It follows that $b^*b = c^*c$ and $bb^* = cc^*$. The equivalences are warranted as the operators are defined on the same subspaces. From the polar decomposition for closed operators [38], the first equality guarantees that $b = U(c^*c)^{1/2}$ for some partial isometry U. Switching roles gives $c = V(b^*b)^{1/2}$ for some partial isometry V. □

5.2 An Energy Ladder Structure for Coupled SUSY

With ladder operators established for coupled SUSY systems, it is natural to inquire about the eigenvalues of the coupled SUSY Hamiltonians a^*a, aa^*, b^*b, and bb^*. As in standard SUSY, a^*a and aa^* share the same eigenvalues—up to a possible eigenvalue of 0. Likewise, b^*b and bb^* share the same eigenvalues—up to a possible eigenvalue of 0. Moreover, the spectra of a^*a and b^*b are related by a shift of γ since $a^*a = b^*b + \gamma$. Thus it is sufficient to study one of a^*a and aa^* to fully understand the eigenvalues of any of the Hamiltonians in a coupled SUSY system.

Theorem 5.8. *If $\{a, b, \gamma, \delta\}$ form an unbroken coupled SUSY system, then the eigenvalues of a^*a are given by $m(\delta - \gamma)$ and $m(\delta - \gamma) + \delta$ where $m \in \mathbb{N}_0$.*

Proof. We first note that if $\psi \in \operatorname{dom} a$ is an eigenfunction of a^*a, then $a^*b\psi \in \operatorname{dom} a$; particularly, it is normalizable. To see this, note that $\langle a^*b\psi, a^*b\psi\rangle = \langle \psi, b^*aa^*b\psi\rangle = \langle \psi, b^*(bb^* + \delta)b\psi\rangle < \infty$ since ψ is also an eigenstate of b^*b. An analogous result holds for aa^*.

Since a^*b is a raising operator for a^*a and 0 is an eigenvalue of a^*a, $m(\delta - \gamma)$ is an eigenvalue for a^*a. Moreover, δ is an eigenvalue for aa^* since 0 is an eigenvalue of bb^*. ba^* is a raising operator for aa^*, so $\delta + m(\delta - \gamma)$ is an eigenvalue for aa^*. Since a^*a and aa^* share eigenvalues—up to 0—$m(\delta - \gamma) + \delta$ is an eigenvalue of a^*a.

These are indeed *all* of the eigenvalues for a^*a. If λ is an eigenvalue of a^*a, then so is $\lambda-(\delta-\gamma)$ since b^*a is a lowering operator for a^*a. If an eigenvalue λ (corresponding to the eigenfunction ψ) existed between 0 and $\delta-\gamma$, then $\lambda-(\delta-\gamma)<0$ would be an eigenvalue of a^*a (corresponding to the eigenfunction $b^*a\psi$) which contradicts the positivity of a^*a. Similar reasoning shows that no eigenvalue can exist between $m(\delta-\gamma)$ and $m(\delta-\gamma)+\delta$, which proves the theorem. $\square$

A consequence of this is that there exist no bounded operator representations for a coupled SUSY system since the eigenvalues are unbounded. Particularly, no finite-dimensional (matricial) representations exist.

With the ladder structure for a^*a (and aa^*) via a^*b and b^*a (and ba^* and ab^*), a richer ladder structure exists than the standard SUSY or QMHO structure. We already know that a and a^* transfer between the sectors generated by a^*a and aa^* so we wish to explore the structure that lies beyond this.

In general, there need not be only one state that is annihilated by a (or more generally two states annihilated by b^*a). For instance, it could be that

$$a=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{d}{dx}+x & 0\\ 0 & \frac{d}{dx}+x\end{pmatrix}, \tag{5.14}$$

which annihilates the states $(\exp(-x^2/2),0)^{\mathrm{T}}$ and $(0,\exp(-x^2/2))^{\mathrm{T}}$. As such we define the following notation.

Definition 5.9. Let $\{a,b,\gamma,\delta\}$ be an unbroken coupled SUSY system. Let $\psi_{i,0}$, $i\in I$ for some finite or at most countable index set I, be an orthonormal family of (normalized) state vectors annihilated by a and $\phi_{j,0}$, $j\in J$ for some finite or at most countable index set J, be an orthonormal family of state vectors annihilated by b^*a but not annihilated by a. Define then $\psi_{i,m} = (a^*b)^m\psi_{i,0}/\|(a^*b)^m\psi_{i,0}\|$ and $\phi_{j,m} = (a^*b)^m\phi_{j,0}/\|(a^*b)^m\phi_{j,0}\|$.

In the case of the harmonic oscillator, the coupled SUSY system collapses because $b = a^*$ and the kernel of a is spanned by the Gaussian h_0, while a vector annihilated by $b^*a = a^2$ but not by a is necessarily a multiple of the first excited state h_1. Note that in general the states annihilated by b^*a but not by a are in one-to-one correspondence with the states annihilated by b^* via the usual SUSY ladder structure.

Consider the following pairs of operators a and b which have the following (infinite) matrix representations in an orthonormal basis:

$$a = \begin{pmatrix} \mathbf{0}_{n\times m} & \sqrt{\delta}\mathbf{1}_{n\times n} & & & \\ & & & & \\ & & \ddots & \sqrt{\delta-\gamma}\mathbf{1}_{m\times m} & \\ & & & \ddots & \ddots \end{pmatrix}, \tag{5.15}$$

$$b = \begin{pmatrix} \mathbf{0}_{n\times m} & & & \\ \sqrt{-\gamma}\mathbf{1}_{m\times m} & & \ddots & \\ & & \sqrt{\delta-\gamma}\mathbf{1}_{n\times n} & \\ & & \ddots & \ddots \end{pmatrix}, \tag{5.16}$$

where $\mathbf{0}_{n_1\times n_2}$ is the $n_1 \times n_2$ zero matrix, $\mathbf{1}_{n_1\times n_2}$ is the $n_1 \times n_2$ ones matrix, and all other elements are taken to be zero. Then taking their adjoints, we have that $a^*a = b^*b + \gamma$ and $aa^* = bb^* + \delta$, however $\dim\ker a \neq \dim\ker b^*$ unless $m = n$, thus $|I| \neq |J|$ in general. Moreover, this shows that one can *build* coupled SUSY systems without much difficulty.

Theorem 5.10. *Let $\{a, b, \gamma, \delta\}$ define an unbroken coupled SUSY system. For any $i \in I$, $j \in J$, and $m \in \mathbb{N}_0$, $\psi_{i,m}$ and $\phi_{j,m}$ are eigenfunctions of a^*a and any normalized eigenfunction of a^*a is of the form $\sum_i \lambda_i \psi_{i,m}$ or $\sum_j \eta_j \phi_{j,m}$ for some finite collection $\lambda_i, \eta_j \in \mathbb{C}$.*

Proof. It is clear by earlier arguments that $\psi_{i,m}, \phi_{j,m}$ are eigenfunctions of a^*a. Conversely, every eigenfunction is linear combination of the $\psi_{i,m}$ or $\phi_{j,m}$ for fixed m. To prove this, we proceed by induction on the eigenvalue of a given eigenfunction ζ.

Without loss of generality, we consider the case that $a^*a\zeta = m(\delta - \gamma)\zeta$. The case of $a^*a\zeta = (m(\delta - \gamma) + \delta)\zeta$ proceeds similarly. For $m = 0$ this is trivial since $\ker a = \operatorname{span}\{\psi_{i,0} : i \in I\}$ and $\ker a^* = \{0\}$. Assume that $a^*a\zeta = m(\delta - \gamma)\zeta$ implies that $\zeta = \sum_i \lambda_i \psi_{i,m}$ for some finite collection $\lambda_i \in \mathbb{C}$. Suppose then that $a^*a\widetilde{\zeta} = (m+1)(\delta - \gamma)\widetilde{\zeta}$. We wish to show that $\widetilde{\zeta} = \sum_i \widetilde{\lambda}_i \psi_{i,m+1}$ for some $\widetilde{\lambda}_i \in \mathbb{C}$.

Applying b^*a to $\widetilde{\zeta}$ yields a state with eigenvalue $m(\delta - \gamma)$ and so $b^*a\zeta = \sum_i \lambda_i \psi_{i,m}$ for some $\lambda_i \in \mathbb{C}$ by the inductive hypothesis. Applying then a^*b we have that

$$\begin{aligned} a^*b \sum \lambda_i \psi_{i,m} &= a^*bb^*a\widetilde{\zeta} \\ &= a^*(aa^* - \delta)a\widetilde{\zeta} \\ &= ((a^*a)^2 - \delta a^*a)\widetilde{\zeta} \\ &= ((m+1)^2(\delta - \gamma)^2 - (m+1)\delta(\delta - \gamma))\widetilde{\zeta}. \end{aligned}$$

Because $\gamma \leq 0$ and $m \geq 0$, $(\delta - \gamma)(m+1) - \delta$ is never zero. Thus $\widetilde{\zeta} = \sum_i \widetilde{\lambda}_i \psi_{i,m+1}$ for some $\widetilde{\lambda}_i$ as claimed. $\square$

As noted above, there is a correspondence between eigenfunctions of a^*a and aa^* via a and a^* in typical SUSY fashion. Thus we can write the eigenfunctions of aa^* as $\widetilde{\psi}_{i,m} = a\psi_{i,m}/\|a\psi_{i,m}\|$ (where $m \neq 0$ since a annihilates $\psi_{i,0}$) and $\widetilde{\phi}_{j,m} = a\phi_{j,m}/\|a\phi_{j,m}\|$. As noted above, the states $\widetilde{\phi}_{j,0}$ are annihilated by b^*. The following figures summarize the coupled SUSY ladder structure.

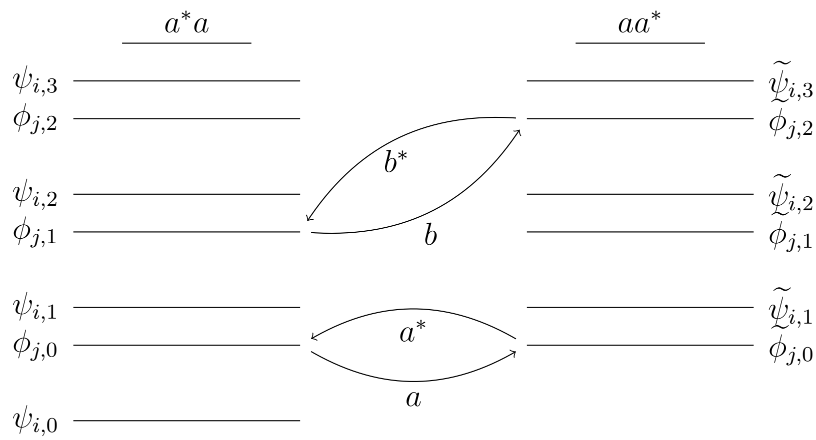

Figure 5.1: The actions of a, b, a^*, and b^* in a coupled SUSY system.

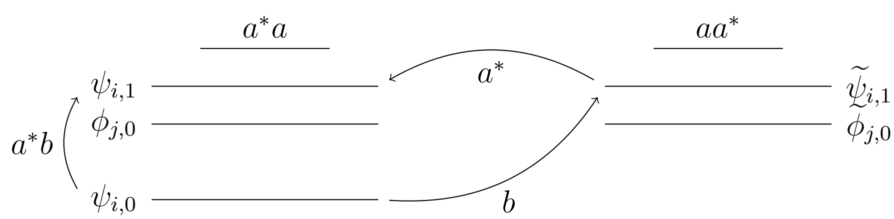

Figure 5.2: The raising operator structure for the first sector in a coupled SUSY system.

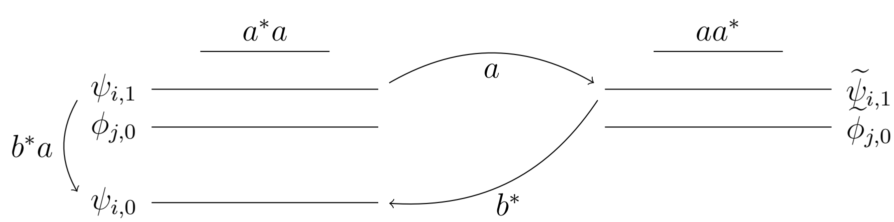

Figure 5.3: The lowering operator structure for the first sector in a coupled SUSY system.

Example 5.11. Returning to the family in Example 1, a simple inductive proof shows that, for a fixed $n \in \mathbb{N}$, the functions

$$\psi_{2m}^{(n)}(x) = e^{\frac{x^{2n}}{2n}} \left(\frac{d}{dx} \frac{1}{x^{2n-2}} \frac{d}{dx} \right)^m e^{-\frac{x^{2n}}{n}} \tag{5.17}$$

$$\psi_{2m+1}^{(n)}(x) = e^{\frac{x^{2n}}{2n}} \left(\frac{d}{dx} \frac{1}{x^{2n-2}} \frac{d}{dx} \right)^m \left(x^{2n-1} e^{-\frac{x^{2n}}{n}} \right). \tag{5.18}$$

are an orthogonal collection of eigenfunctions of $a_n^* a_n$ satisfying $\psi_{m+2}^{(n)} = \lambda_{m,n} a_n^* b_n \psi_m^{(n)}$ for some $\lambda_{m,n}$. It is not hard to see that these are polynomials multiplying $e^{-\frac{x^{2n}}{2n}}$ and analysis similar to that in Chapter 3 shows that they form a basis for $L^2(\mathbb{R})$. In the case of $n = 1$, these become the usual Hermite functions up to normalization. Similar relations hold for the eigenfunctions of $a_n a_n^*$.

5.3 Coupled SUSY and Other Oscillator Systems

Traditionally the QMHO is associated to the 1D Heisenberg-Weyl Lie algebra as this is the Lie algebra which corresponds to the canonical commutation relations which is reflected in the algebra generated by the ladder operators. This is not the only Lie algebra which may be associated to the QMHO. There are two other treatments of the QMHO: Schwinger's "spinification" of the two-particle QMHO and the $\mathfrak{su}(1,1)$ treatment of the QMHO. Coupled SUSY is to some degree a unification of the two treatments.

In Schwinger's "spinification" of the QMHO [46], one considers two independent oscillators and defines the operators

$$\mathcal{Q}_\nu = \sum_{i,j=1}^{2} \frac{1}{2} a_i^* (\sigma_\nu)^{ij} a_j, \tag{5.19}$$

where for $\nu = 1,2,3$, σ_ν is the νth Pauli matrix. Explicitly, $\mathcal{Q}_1 = \frac{1}{2}(a_1^* a_2 + a_1 a_2^*)$, $\mathcal{Q}_2 = -\frac{i}{2}(a_1^* a_2 - a_1 a_2^*)$, and $\mathcal{Q}_3 = \frac{1}{2}(a_1^* a_1 - a_2^* a_2)$. The operators $\mathcal{Q}_\nu$ form the $\mathfrak{su}(2)$ Lie algebra, for instance

$$\begin{aligned}
[\mathcal{Q}_1, \mathcal{Q}_2] &= -\frac{i}{4}[a_1^* a_2 + a_1 a_2^*, a_1^* a_2 - a_1 a_2^*] \\
&= \frac{i}{2}[a_1^* a_2, a_1 a_2^*] \\
&= \frac{i}{2}(a_1^* a_1 a_2 a_2^* - a_1 a_1^* a_2^* a_2) \\
&= \frac{i}{2}(a_1^* a_1 - a_2^* a_2) \\
&= i\mathcal{Q}_3.
\end{aligned}$$

The finite dimensional representations in this system have a fixed quantum number which corresponds to the energy difference between the two oscillators and the individual spin states for a fixed energy correspond to the different configurations within each energy level. The Lie algebra $\mathfrak{su}(2)$ is equipped with its own ladder operators. In this case, defining $\mathcal{Q}_\pm = \mathcal{Q}_1 \pm i\mathcal{Q}_2$, we have

$$[\mathcal{Q}_3, \mathcal{Q}_\pm] = \pm\mathcal{Q}_\pm. \tag{5.20}$$

A simple computation shows that $\mathcal{Q}_+ = a_1^* a_2$ and $\mathcal{Q}_- = a_2^* a_1$. These are analogous to the (quadratic) ladder operators for a coupled SUSY system.

Similar to Schwinger's $\mathfrak{su}(2)$ "spinification" of the QMHO is an $\mathfrak{su}(1,1)$ representation of the QMHO [36]. If a, a^* represent the usual QMHO ladder operators, then letting $\mathcal{K}_0 = a^* a + \frac{1}{2}$, $\mathcal{K}_+ = (a^*)^2$, and $\mathcal{K}_- = a^2$, we have that

$$[\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \quad [\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0, \tag{5.21}$$

which is exactly the $\mathfrak{su}(1,1)$ Lie algebra.

Coupled SUSY has elements of both of the above approaches to the QMHO. The ladder operators in all three cases are second order; in coupled SUSY, the ladder operators are a combination of the operators coming from two separate Hamiltonians; and coupled SUSY retains the $\mathfrak{su}(1,1)$ structure implicitly built into the QMHO.

5.4 Coherent States for Coupled SUSY Systems

In traditional SUSY, coherent states have been developed, however the coherent states do not mimic those of the QMHO even though they do exploit the SUSY structure [4, 13, 16]. Particularly, QMHO coherent states are often taken to be eigenfunctions of the lowering operator; SUSY coherent states do not enjoy this property or a property similar to it. Part of the reason for this is the lack of a true ladder structure in SUSY. As seen above, coupled SUSY has a rich ladder structure that has elements of both SUSY and the QMHO. This suggests that the coherent states for a coupled SUSY system should behave somewhat analogously to that of the QMHO while retaining a SUSY flavor.

In general, there are several kinds of coherent states, depending on the context in which one is interested. Coherent states can be seen to be uncertainty minimizers, eigenstates of a lowering operator, specific infinite series of basis functions, or generalized displacements of a cyclic vector. In the case of the QMHO, these are all the same, however in general this is not the case. For $\mathfrak{su}(1,1)$, the various forms of coherent states have been studied extensively [9, 23, 36]. We elect to use the displacement operator definition for coherent states as it uses the Lie algebraic properties of $\mathfrak{su}(1,1)$ and ties well into the coupled SUSY formalism as there are natural cyclic vectors in $\psi_{i,0}$, $\phi_{j,0}$, $\widetilde{\psi}_{i,1}$, and $\widetilde{\phi}_{j,0}$.

If one has $\mathfrak{su}(1,1)$ operators $\mathcal{K}_0, \mathcal{K}_\pm$ where

$$[\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \quad [\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0, \tag{5.22}$$

then the operator $\mathcal{D}(z) = \exp(z\mathcal{K}_+ - \bar{z}\mathcal{K}_-)$ defines an $\mathfrak{su}(1,1)$ displacement operator [36, p. 74]. Suppose that $\mathcal{K}_0\psi_m = (m+k)\psi_m$ for a basis of states ψ_m, where $\psi_{m+1} = \mathcal{K}_+\psi_m/\|\mathcal{K}_+\psi_m\|$. The coherent state generated by $\mathcal{D}(z)$ can be written as

$$|z;k\rangle = \mathcal{D}(z)\psi_0 = (1-|z|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{\frac{1}{2}} z^m \psi_m. \tag{5.23}$$

Suppose that $\{a, b, \gamma, \delta\}$ defines a coupled SUSY system. Define then the following operators

$$\mathcal{K}_0 = \frac{1}{\delta-\gamma}\left(a^*a - \frac{\gamma}{2}\right), \quad \mathcal{K}_+ = \frac{1}{\delta-\gamma}a^*b, \quad \mathcal{K}_- = \frac{1}{\delta-\gamma}b^*a, \tag{5.24}$$

$$\widetilde{\mathcal{K}_0} = \frac{1}{\delta-\gamma}\left(aa^* - \frac{\delta}{2}\right), \quad \widetilde{\mathcal{K}_+} = \frac{1}{\delta-\gamma}ba^*, \quad \widetilde{\mathcal{K}_-} = \frac{1}{\delta-\gamma}ab^*. \tag{5.25}$$

A straightforward calculation shows that the relations in (5.22) hold. Since we have four families of cyclic vectors (indexed by i and j), we have four sets of coherent states (indexed by i and j). A simple computation shows that

$$\mathcal{K}_0\psi_{i,0} = -\frac{\gamma}{2(\delta-\gamma)}\psi_{i,0} \tag{5.26}$$

$$\mathcal{K}_0\phi_{j,0} = \left(\frac{\delta}{2(\delta-\gamma)} + \frac{1}{2}\right)\phi_{j,0} \tag{5.27}$$

$$\widetilde{\mathcal{K}_0}\widetilde{\psi}_{i,1} = \left(-\frac{\gamma}{2(\delta-\gamma)} + \frac{1}{2}\right)\widetilde{\psi}_{i,1} \tag{5.28}$$

$$\widetilde{\mathcal{K}_0}\widetilde{\phi}_{j,0} = \frac{\delta}{2(\delta-\gamma)}\widetilde{\phi}_{j,0}, \tag{5.29}$$

from which we get the following coherent states:

$$\left|z; -\frac{\gamma}{2(\delta-\gamma)}\right\rangle_i = (1-|z|^2)^{-\frac{\gamma}{2(\delta-\gamma)}} \sum_{m=0}^{\infty} \left(\frac{\Gamma\left(m-\frac{\gamma}{\delta-\gamma}\right)}{m!\,\Gamma\left(-\frac{\gamma}{\delta-\gamma}\right)} \right)^{\frac{1}{2}} z^m \psi_{i,m} \tag{5.30}$$

$$\left|z; \frac{\delta}{2(\delta-\gamma)}+\frac{1}{2}\right\rangle_j = (1-|z|^2)^{\frac{\delta}{2(\delta-\gamma)}+\frac{1}{2}} \sum_{m=0}^{\infty} \left(\frac{\Gamma\left(m+\frac{\delta}{\delta-\gamma}+1\right)}{m!\,\Gamma\left(\frac{\delta}{\delta-\gamma}+1\right)} \right)^{\frac{1}{2}} z^m \phi_{j,m} \tag{5.31}$$

$$\left|z; -\frac{\gamma}{2(\delta-\gamma)}+\frac{1}{2}\right\rangle_i = (1-|z|^2)^{-\frac{\gamma}{2(\delta-\gamma)}+\frac{1}{2}} \sum_{m=1}^{\infty} \left(\frac{\Gamma\left(m-\frac{\gamma}{\delta-\gamma}+1\right)}{m!\,\Gamma\left(-\frac{\gamma}{\delta-\gamma}+1\right)} \right)^{\frac{1}{2}} z^m \widetilde{\psi}_{i,m} \tag{5.32}$$

$$\left|z; \frac{\delta}{2(\delta-\gamma)}\right\rangle_j = (1-|z|^2)^{\frac{\delta}{2(\delta-\gamma)}} \sum_{m=0}^{\infty} \left(\frac{\Gamma\left(m+\frac{\delta}{\delta-\gamma}\right)}{m!\,\Gamma\left(\frac{\delta}{\delta-\gamma}\right)} \right)^{\frac{1}{2}} z^m \widetilde{\phi}_{j,m} \tag{5.33}$$

In the case of the QMHO, the coherent states are eigenstates of the lowering operator. Since coupled SUSY behaves so similarly to the QMHO, it is natural to ask what happens to the coupled SUSY coherent states under an application of the lowering operators. The lowering operators are composed of a and b^*, so we want to investigate the action of a and b^* on the coherent states. To this end, we have the following lemma.

Lemma 5.12. *Let $\psi_{i,m}$, $\phi_{j,m}$, $\widetilde{\phi}_{i,m}$, and $\widetilde{\phi}_{j,m}$ be as above. Then*

$$a\psi_{i,m} = \sqrt{m(\delta-\gamma)}\,\widetilde{\psi}_{i,m} \tag{5.34}$$

$$a\phi_{j,m} = \sqrt{(\delta-\gamma)\left(m+\frac{\delta}{\delta-\gamma}\right)}\,\widetilde{\phi}_{j,m} \tag{5.35}$$

$$b^* \widetilde{\psi}_{i,m} = \sqrt{(\delta - \gamma)\left(m - \frac{\delta}{\delta - \gamma}\right)} \psi_{i,m-1} \tag{5.36}$$

$$b^* \widetilde{\phi}_{j,m} = \sqrt{m(\delta - \gamma)} \phi_{j,m-1} \tag{5.37}$$

Proof. We only prove the first as the others follow in exactly the same manner. We assume without loss of generality that the proportionality is pure real as a global phase does not change the underlying mathematics. By definition, $a\psi_{i,m} = \lambda \widetilde{\psi}_{i,m}$, so we need only to solve for λ.

$$\begin{aligned} |\lambda|^2 &= \langle a\psi_{i,m}, a\psi_{i,m} \rangle \\ &= \langle a^* a \psi_{i,m}, \psi_{i,m} \rangle \\ &= m(\delta - \gamma). \end{aligned}$$

The last equality follows since $\psi_{i,m}$ is an eigenfunction of a^*a with eigenvalue $m(\delta - \gamma)$. Thus $\lambda = \sqrt{m(\delta - \gamma)}$ as desired. □

Because our ladder operators can only relate $\psi_{i,m}$ with $\widetilde{\psi}_{i,m'}$ and $\phi_{j,m}$ with $\widetilde{\phi}_{j,m'}$ (and vice versa), it is clear that we can only relate (5.30) with (5.32) and (5.31) with (5.33) (and vice versa). Making use of relations (5.34)-(5.37), we have that

$$a \left| z; -\frac{\gamma}{2(\delta - \gamma)} \right\rangle_i = \sqrt{-\gamma} \frac{z}{\sqrt{1 - |z|^2}} \left| z; -\frac{\gamma}{2(\delta - \gamma)} + \frac{1}{2} \right\rangle_i, \tag{5.38}$$

$$b^* \left| z; \frac{\delta}{2(\delta - \gamma)} \right\rangle_j = \sqrt{\delta} z \sqrt{1 - |z|^2} \left| z; -\frac{\delta}{2(\delta - \gamma)} + \frac{1}{2} \right\rangle_j. \tag{5.39}$$

If we were to apply b^* to the first or a to the second, we would not retain a multiple of the original state as the relations (5.34)-(5.37) indicate. Thus, while the coherent states are not eigenstates of the lowering operators b^*a or ab^*, applying half of one of the lowering operators can convert one coherent state into another—up to a multiplicative factor (since the operators are not unitary). This appears to be a new structure in SUSY and QMHO coherent states.

5.5 Uncertainty Principles for Coupled SUSY Systems

The canonical uncertainty principle in quantum mechanics is the Heisenberg uncertainty principle which is an uncertainty principle between the position operator x and the momentum operator p. The Heisenberg uncertainty principle says that, in natural units, the standard deviation in position and momentum is bounded below by $\sigma_x \sigma_p \geq \frac{1}{2}$.

The minimizer of the uncertainty principle is the Gaussian (and translations and modulations thereof). This is easily proved via Cauchy-Schwarz techniques [26]. Since x and p can be written as linear combinations of the QMHO ladder operators (i.e., $x = \frac{1}{\sqrt{2}}(a + a^*)$ and $p = \frac{i}{\sqrt{2}}(a - a^*)$), we expect to realize uncertainty principles for coupled SUSY systems in a similar way via their ladder operators.

Definition 5.13. Let $\{a, b, \gamma, \delta\}$ define a coupled SUSY system. We define the following analogues of the traditional position and momentum operators for the separate sectors:

$$\mathcal{L} = -\frac{1}{2}(a^*b + b^*a), \quad \mathcal{A} = \frac{i}{2}(a^*b - b^*a), \tag{5.40}$$

$$\widetilde{\mathcal{L}} = -\frac{1}{2}(ba^* + ab^*), \quad \widetilde{\mathcal{A}} = \frac{i}{2}(ba^* - ab^*). \tag{5.41}$$

In the case of the family of examples in Example 1, $\mathcal{L} = -\frac{1}{2}\frac{d}{dx}\frac{1}{x^{2n-2}}\frac{d}{dx} - \frac{1}{2}x^{2n}$ resembles a Lagrangian and is exactly a Lagrangian operator $-\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2$ when $n = 1$, whereas $\mathcal{A} = \frac{1}{2}\{x, p\}$ is precisely the dilation operator.

Theorem 5.14. *Let $\{a, b, \gamma, \delta\}$ be an unbroken coupled SUSY system, $\mathcal{L}$ and $\mathcal{A}$ be as above, and, also as above,* $\ker a = \{\psi_{i,0} : i \in I\}$ *for some index set I. An uncertainty principle holds for $\mathcal{L}$ and $\mathcal{A}$ and the minimizers are the states $\psi_{i,0}$.*

Proof. Let ψ be a normalized wavefunction. Note that Robertson's uncertainty relation [22, p.53] gives us that

$$\begin{aligned}(\sigma_{\mathcal{L}}\sigma_{\mathcal{A}})_\psi &\geq \frac{1}{2}|\langle\psi|[\mathcal{L}, \mathcal{A}]|\psi\rangle| \\ &= \frac{1}{4}|\langle\psi|[a^*b, b^*a] - [b^*a, a^*b]|\psi\rangle| \\ &= \frac{\delta - \gamma}{4}|2\langle\psi|a^*a|\psi\rangle - \gamma|.\end{aligned}$$

Because $\gamma < 0$, a has annihilating states $\psi_{i,0}$, and a^*a is self-adjoint, the lower bound given by these states is

$$\sigma_{\mathcal{L}}\sigma_{\mathcal{P}} \geq \frac{1}{4}(\delta - \gamma)|\gamma|. \tag{5.42}$$

This does not guarantee that this lower bound is indeed attained. For the states $\psi_{i,0}$, we have

$$\begin{aligned}\langle\mathcal{L}\rangle &= \langle\phi_{i,0}|\mathcal{L}|\psi_{i,0}\rangle \\ &= \frac{1}{2}\langle\phi_{i,0}|a^*b + b^*a|\psi_{i,0}\rangle \\ &= 0.\end{aligned}$$

Similarly, $\langle\mathcal{A}\rangle = 0$. Evaluating $\langle\mathcal{L}^2\rangle$, making use of the fact that a annihilates the states $\psi_{i,0}$, and employing the coupled SUSY structure, it follows that

$$\begin{aligned}\langle \mathcal{L}^2 \rangle &= \frac{1}{4}\langle \psi_{i,0}|(b^*a + a^*b)^2|\psi_{i,0}\rangle \\ &= \frac{1}{4}\langle \psi_{i,0}|b^*aa^*b|\psi_{i,0}\rangle \\ &= \frac{1}{4}\langle \psi_{i,0}|b^*(bb^* + \delta)b|\psi_{i,0}\rangle \\ &= \frac{1}{4}\langle \psi_{i,0}|(a^*a - \gamma)^2 + \delta(a^*a - \gamma)|\psi_{i,0}\rangle \\ &= \frac{1}{4}(\delta - \gamma)|\gamma|.\end{aligned}$$

An identical result holds for $\langle \mathcal{A}^2 \rangle$, thus we have that

$$\begin{aligned}\sigma_{\mathcal{L}}\sigma_{\mathcal{A}} &= \sqrt{\langle \mathcal{L}^2\rangle - \langle \mathcal{L}\rangle^2}\sqrt{\langle \mathcal{A}^2\rangle - \langle \mathcal{A}\rangle^2} \\ &= \frac{1}{4}(\delta - \gamma)|\gamma|.\end{aligned}$$

The states $\psi_{i,0}$ are indeed the *only* minimizers. If $a\psi \neq 0$, then $\langle \psi|a^*a|\psi\rangle > 0$. Because $\gamma \leq 0$, it follows that the uncertainty product is strictly greater than $\frac{1}{4}(\delta - \gamma)|\gamma|$. □

Theorem 5.15. *Let $\{a, b, \gamma, \delta\}$ define an unbroken coupled SUSY system, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{A}}$ be as above, and, also as above,* $\ker b^* = \{\widetilde{\phi}_{j,0} : j \in J\}$ *for some index set J. An uncertainty principle holds for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{A}}$ and the minimizers are the states $\widetilde{\phi}_{j,0}$.*

The proof of this theorem is nearly identical to that of the previous one so we omit it, however the uncertainty principle is now given by

$$\sigma_{\widetilde{\mathcal{L}}}\sigma_{\widetilde{\mathcal{A}}} \geq \frac{1}{4}(\delta - \gamma)\delta. \tag{5.43}$$

Often in SUSY one treats the first and second sectors simultaneously in a matrix formulation by defining the operators $\mathcal{H}$, $\mathcal{Q}$, and $\mathcal{Q}^*$ which act on the direct sum of the two sectors as follows:

$$\mathbb{H} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix}, \quad \mathbb{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{Q}_1 & 0 \end{pmatrix}, \quad \mathbb{Q}^* = \begin{pmatrix} 0 & \mathcal{Q}_1^* \\ 0 & 0 \end{pmatrix}. \tag{5.44}$$

The logic being that the joint Hamiltonian should act on the two subspaces separately by their own Hamiltonians and is therefore diagonal, and the joint charge operators should be off-diagonal because $\mathcal{Q}_1$ and $\mathcal{Q}_1^*$ transfer between the two sectors.

This allows us to define a tertiary set of first order position and momentum operators. Previously, the analysis was relegated to second order position and momentum operators because the operators $\mathcal{L}$, $\mathcal{A}$, $\widetilde{\mathcal{L}}$, and $\widetilde{\mathcal{A}}$ acted within a sector; however by combining the two sectors into one framework, we allow ourselves the ability to drop down to first order operators, analogous to the usual QMHO case.

Definition 5.16. Again, let $\{a, b, \gamma, \delta\}$ define a coupled SUSY system. We define the operators $\mathcal{X}$ and $\mathcal{P}$ on the direct sum of the two sectors as follows:

$$\mathbb{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & a^* + b^* \\ a + b & 0 \end{pmatrix}, \quad \mathbb{P} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & a^* - b^* \\ -a + b & 0 \end{pmatrix}. \tag{5.45}$$

For the infinite family of operators in Example 1, $a_n + b_n$ yields $\sqrt{2}x^n$ and $a_n - b_n$ yields $\frac{\sqrt{2}}{x^{n-1}} \frac{d}{dx}$. Hence $a_n + b_n$ extracts the coordinate-like object corresponding to the coupled SUSY system, whereas $a_n - b_n$ extracts the derivative-like object corresponding to the coupled SUSY system. Therefore, $\mathbb{X}$ plays the role of a generalized notion of position and $\mathbb{P}$ plays the role of a generalized notion of momentum [13].

Theorem 5.17. *Let $\{a, b, \gamma, \delta\}$ define an unbroken coupled SUSY system and $\mathbb{X}$ and $\mathbb{P}$ be as above, then the following uncertainty principle holds for $\mathbb{X}$ and $\mathbb{P}$:*

$$\sigma_{\mathbb{X}}\sigma_{\mathbb{P}} \geq \frac{1}{2}\min\{|\gamma|, \delta\}. \tag{5.46}$$

Proof. Let $\Psi = (\psi_1, \psi_2)^{\mathrm{T}}$ be the state in which we are evaluating the expectation, then $1 = \|\psi_1\|^2 + \|\psi_2\|^2$. Again making use of Robertson's inequality, we have that

$$\begin{aligned}
(\sigma_{\mathbb{X}}\sigma_{\mathbb{P}})_\Psi &\geq \frac{1}{2}|\langle\Psi|[\mathbb{X}, \mathbb{P}]|\Psi\rangle| & (5.47)\\
&= \frac{1}{4}\left|\left\langle \Psi \left| \left[\begin{pmatrix} 0 & a^* + b^* \\ a + b & 0 \end{pmatrix}, \begin{pmatrix} 0 & a^* - b^* \\ -a + b & 0 \end{pmatrix} \right] \right| \Psi \right\rangle\right| & (5.48)\\
&= \frac{1}{4}\left|\left\langle \Psi \left| \begin{pmatrix} -2\gamma & 0 \\ 0 & 2\delta \end{pmatrix} \right| \Psi \right\rangle\right| & (5.49)\\
&= \frac{1}{2}(|\gamma|\|\psi_1\|^2 + \delta\|\psi_2\|^2). & (5.50)
\end{aligned}$$

Since $\|\psi_2\|^2 = 1 - \|\psi_1\|^2$, the above is a convex combination of $|\gamma|$ and δ, so indeed we have that

$$\sigma_{\mathbb{X}}\sigma_{\mathbb{P}} \geq \frac{1}{2}\min\{|\gamma|, \delta\}. \tag{5.51}$$

We now show that the value of $\frac{1}{2}|\gamma|$ is attainable. Let $\Psi = (\psi_{i,0}, 0)^{\mathrm{T}}$, where $\psi_{i,0}$ is as above. The case of $\frac{1}{2}\delta$ proceeds similarly by taking $\Psi = (0, \widetilde{\phi}_{j,0})^{\mathrm{T}}$, where $\widetilde{\phi}_{j,0}$ is also as above. For this choice of Ψ, it follows that

$$\langle\Psi|\mathbb{X}|\Psi\rangle = \frac{1}{\sqrt{2}}(\langle\psi_{i,0}|a^* + b^*|0\rangle + \langle 0|a + b|\psi_{i,0}\rangle) = 0, \tag{5.52}$$

$$\langle\Psi|\mathbb{P}|\Psi\rangle = -\frac{i}{\sqrt{2}}(\langle\psi_{i,0}|a^* - b^*|0\rangle + \langle 0| - a + b|\psi_{i,0}\rangle) = 0. \tag{5.53}$$

Computing $\mathbb{X}^2$ and $\mathbb{P}^2$ yields

$$\mathbb{X}^2 = \frac{1}{2}\begin{pmatrix} (a^*+b^*)(a+b) & 0 \\ 0 & (a+b)(a^*+b^*) \end{pmatrix} \tag{5.54}$$

$$= \frac{1}{2}\begin{pmatrix} a^*a + a^*b + b^*a + b^*b & 0 \\ 0 & aa^* + ab^* + ba^* + bb^* \end{pmatrix} \tag{5.55}$$

$$\mathbb{P}^2 = \frac{1}{2}\begin{pmatrix} (a^*-b^*)(a-b) & 0 \\ 0 & (a-b)(a^*-b^*) \end{pmatrix} \tag{5.56}$$

$$= \frac{1}{2}\begin{pmatrix} a^*a - a^*b - b^*a + b^*b & 0 \\ 0 & aa^* - ab^* - ba^* + bb^* \end{pmatrix} \tag{5.57}$$

Inspecting the diagonal terms, it is clear that this uncertainty principle is quite different from that of $\mathcal{L}$ and $\mathcal{A}$ (and from that of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{A}}$). We are only concerned with the upper left elements since we are considering states of the form $\Psi = (\psi_{i,0}, 0)^{\mathrm{T}}$. Noting that a annihilates $\psi_{i,0}$ and using $b^*b = a^*a - \gamma$, it follows that

$$\langle \Psi | \mathbb{X}^2 | \Psi \rangle = -\frac{1}{2}\gamma = \langle \Psi | \mathbb{P}^2 | \Psi \rangle. \tag{5.58}$$

Since $\gamma \leq 0$, the result follows. □

Remark 5.18. *The above uncertainty principles agree exactly with the traditional Heisenberg uncertainty principle in the case of the QMHO since $\gamma = -1$ and $\delta = 1$, giving an uncertainty bound of $\frac{1}{2}$ for each with the minimizers being Gaussians. For $n > 1$, the uncertainty product $\sigma_{\mathcal{X}}\sigma_{\mathcal{P}}$ has a lower bound of $\frac{1}{2}$, just as in the Heisenberg uncertainty principle with minimizers $\exp(-x^{2n}/2n)$, but the uncertainty products $\sigma_{\mathcal{L}}\sigma_{\mathcal{A}}$ and $\sigma_{\widetilde{\mathcal{L}}}\sigma_{\widetilde{\mathcal{A}}}$ are for $n > 1$ bounded by a larger constant since $\delta - \gamma = 2n$.*

5.6 A Family of Coupled SUSY Integral Transforms

The normalized eigenfunctions of a^*a were denoted by $\psi_{i,m}$ and $\phi_{j,m}$ and the normalized eigenfunctions for aa^* were denoted by $\widetilde{\psi}_{i,m}$ and $\widetilde{\phi}_{j,m}$ for some $i \in I$, $j \in J$ and m. It is not clear that the coupled SUSY algebra is sufficient in determining the completeness of the eigenfunctions of a^*a or aa^*. To this end, define the following Hilbert subspaces for fixed i, j:

$$\mathfrak{H}_i^+ := \overline{\operatorname{span}}\{\psi_{i,m} : m \in \mathbb{N}_0\}, \tag{5.59}$$

$$\mathfrak{H}_j^- := \overline{\operatorname{span}}\{\phi_{j,m} : m \in \mathbb{N}_0\}, \tag{5.60}$$

$$\widetilde{\mathfrak{H}}_i^+ := \overline{\operatorname{span}}\{\widetilde{\psi}_{i,m} : m \in \mathbb{N}\}, \tag{5.61}$$

$$\widetilde{\mathfrak{H}}_j^- := \overline{\operatorname{span}}\{\widetilde{\phi}_{j,m} : m \in \mathbb{N}_0\} \tag{5.62}$$

Assume that the parent Hilbert space is realized as an $L^2(\Omega)$ space for some measure space Ω. Define the L^2 distributions c_i, s_j, $\widetilde{c}_i$, and $\widetilde{s}_j$ by

$$c_i(x,y) = \sum_{m=0}^{\infty}(-1)^m \psi_{i,m}(x)\psi_{i,m}(y), \tag{5.63}$$

$$s_j(x,y) = -\sum_{m=0}^{\infty}(-1)^m \phi_{j,m}(x)\phi_{j,m}(y), \tag{5.64}$$

$$\widetilde{c}_i(x,y) = \sum_{m=1}^{\infty}(-1)^m \widetilde{\psi}_{i,m}(x)\widetilde{\psi}_{i,m}(y), \tag{5.65}$$

$$\widetilde{s}_j(x,y) = -\sum_{m=0}^{\infty}(-1)^m \widetilde{\phi}_{j,m}(x)\widetilde{\phi}_{j,m}(y). \tag{5.66}$$

The functions c_i, etc., can be thought of as elements of a rigged Hilbert space [3].

Particularly, taking $\mathfrak{X}_i^+ \subseteq L^2(\Omega)$ to be $\operatorname{span}\{\psi_{i,m} : m \in \mathbb{N}_0\}$, then the linear maps $y \mapsto \langle f, c_i(\cdot, y)\rangle$ are bounded. Thus c_i, etc., define linear functionals on $\mathfrak{X}_i^+$, etc., that is they can be realized as elements of $(\mathfrak{X}_i^+)^*$, etc., and can be considered L^2 "distributions."

The above equalities are to be understood in the weak sense, e.g., for sufficiently nice $f \in \mathfrak{H}_i^+$, we have

$$\langle f, c_i(\cdot, y)\rangle := \sum_{m=0}^{\infty} (-1)^m \psi_{i,m}(y) \langle f, \psi_{i,m}\rangle. \tag{5.67}$$

To this end, define the integral operators Φ_i^+, Φ_j^-, $\widetilde{\Phi}_i^+$, and $\widetilde{\Phi}_j^-$ by

$$\Phi_i^+ f_1(y) := \langle f_1, c_i(\cdot, y)\rangle, \tag{5.68}$$

$$\Phi_j^- f_2(y) := \langle f_2, s_j(\cdot, y)\rangle, \tag{5.69}$$

$$\widetilde{\Phi}_i^+ f_3(y) := \langle f_3, \widetilde{c}_i(\cdot, y)\rangle, \tag{5.70}$$

$$\widetilde{\Phi}_j^- f_4(y) := \langle f_4, \widetilde{s}_j(\cdot, y)\rangle. \tag{5.71}$$

The integral operators are well-defined on the bases generated by the eigenfunctions of the coupled SUSY Hamiltonians and have eigenvalues ± 1. It is a straightforward calculation to show that the coupled SUSY eigenfunctions are also eigenfunctions of the integral operators (in the appropriate ways). As such, the integral operators extend to unitaries on their respective Hilbert spaces.

The integral operators are intertwined and skew-intertwined via the coupled SUSY charge operators, that is

$$a\Phi_i^+ = \widetilde{\Phi}_i^+ a \qquad\qquad \Phi_i^+ a^* = a^* \widetilde{\Phi}_i^+ \tag{5.72}$$

$$a\Phi_j^- = \widetilde{\Phi}_j^- a \qquad \Phi_j^- a^* = a^* \widetilde{\Phi}_j^- \tag{5.73}$$

$$b\Phi_i^+ = -\widetilde{\Phi}_i^+ b \qquad \Phi_i^+ b^* = -b^* \widetilde{\Phi}_i^+ \tag{5.74}$$

$$b\Phi_j^- = -\widetilde{\Phi}_j^- b \qquad \Phi_j^- b^* = -b^* \widetilde{\Phi}_j^- \tag{5.75}$$

This is easily proved by considering basis elements and making use of the fact that the charge operators are, by assumption, closed, e.g.,

$$a\Phi_i^+ \psi_{i,m'}(y) = a \sum_{m=0}^{\infty} (-1)^m \psi_{i,m} \langle \psi_{i,m'}, \psi_{i,m} \rangle \tag{5.76}$$

$$= a \sum_{m=0}^{\infty} (-1)^m \delta_{m,m'} \psi_{i,m} \tag{5.77}$$

$$= (-1)^{m'} a\psi_{i,m'} \tag{5.78}$$

Noting that $a\psi_{i,m} = \lambda_m \widetilde{\psi}_{i,m}$ for some scalar λ_m by definition, we have that

$$\widetilde{\Phi}_i^+ a\psi_{i,m'}(y) = \langle a\psi_{i,m'}, \widetilde{c}_i(\cdot, y) \rangle \tag{5.79}$$

$$= \sum_{m=1}^{\infty} (-1)^m \widetilde{\psi}_{i,m} \langle a\psi_{i,m'}, \widetilde{\psi}_{i,m} \rangle \tag{5.80}$$

$$= \sum_{m=1}^{\infty} (-1)^m \lambda_m \widetilde{\psi}_{i,m} \langle \widetilde{\psi}_{i,m'}, \widetilde{\psi}_{i,m} \rangle \tag{5.81}$$

$$= \sum_{m=1}^{\infty} (-1)^m \lambda_m \widetilde{\psi}_{i,m} \delta_{m,m'} \tag{5.82}$$

$$= (-1)^{m'} a\psi_{i,m'}. \tag{5.83}$$

The other cases proceed similarly.

These integral operators generalize the usual Fourier sine and cosine transforms. In the harmonic oscillator case, these integral operators are exactly the Fourier sine and cosine transforms. Furthermore, the Φ_n and $\widetilde{\Phi}_n$ transforms are subsumed in this as well by letting $a = \frac{1}{x^{n-1}}\frac{d}{dx} + x^n$ and $b = -\frac{1}{x^{n-1}}\frac{d}{dx}$.

5.7 The Coupled SUSY Algebra

As noted in Chapter 4, the matrix formulation of SUSY carries with it an algebra structure. In coupled SUSY, there is not just one $\mathbb{Q}$ and $\mathbb{Q}^*$ like in traditional SUSY, rather, but two of each: one corresponding to a (or a^*) and one corresponding to b (or b^*). Define then

$$\mathbb{Q}_a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \qquad \mathbb{Q}_b = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \tag{5.84}$$

$$\mathbb{Q}_a^* = \begin{pmatrix} 0 & a^* \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{Q}_b^* = \begin{pmatrix} 0 & b^* \\ 0 & 0 \end{pmatrix}, \tag{5.85}$$

$$\mathbb{H}_a = \begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix}, \qquad \mathbb{H}_b = \begin{pmatrix} b^*b & 0 \\ 0 & bb^* \end{pmatrix} \tag{5.86}$$

The usual SUSY superalgebra is present as expected but there is a richer structure for coupled SUSY. Define the (lowering and raising) operators $\mathbb{L}$ and $\mathbb{L}^*$ by

$$\mathbb{L} = \begin{pmatrix} b^*a & 0 \\ 0 & ab^* \end{pmatrix}, \quad \mathbb{L}^* = \begin{pmatrix} a^*b & 0 \\ 0 & ba^* \end{pmatrix}. \tag{5.87}$$

We have then the following commutation and anti-commutation relations:

$$[\mathbb{H}_a, \mathbb{H}_a] = 0, \qquad [\mathbb{H}_b, \mathbb{H}_b] = 0, \qquad [\mathbb{H}_a, \mathbb{H}_b] = 0, \tag{5.88}$$

$$[\mathbb{H}_a, \mathbb{Q}_a] = 0, \qquad [\mathbb{H}_a, \mathbb{Q}_a^*] = 0, \tag{5.89}$$

$$[\mathbb{H}_b, \mathbb{Q}_a] = -(\delta - \gamma)\mathbb{Q}_a, \qquad [\mathbb{H}_b, \mathbb{Q}_a^*] = (\delta - \gamma)\mathbb{Q}_a^*, \tag{5.90}$$

$$[\mathbb{H}_a, \mathbb{Q}_b] = (\delta - \gamma)\mathbb{Q}_b, \qquad [\mathbb{H}_a, \mathbb{Q}_b^*] = -(\delta - \gamma)\mathbb{Q}_b^*, \tag{5.91}$$

$$[\mathbb{H}_b, \mathbb{Q}_b] = 0, \qquad [\mathbb{H}_b, \mathbb{Q}_b^*] = 0, \tag{5.92}$$

$$[\mathbb{H}_a, \mathbb{L}] = -(\delta - \gamma)\mathbb{L}, \qquad [\mathbb{H}_a, \mathbb{L}^*] = (\delta - \gamma)\mathbb{L}^* \tag{5.93}$$

$$[\mathbb{H}_b, \mathbb{L}] = -(\delta - \gamma)\mathbb{L}, \qquad [\mathbb{H}_b, \mathbb{L}^*] = (\delta - \gamma)\mathbb{L}^* \tag{5.94}$$

$$\{\mathbb{Q}_a, \mathbb{Q}_a\} = 0, \qquad \{\mathbb{Q}_a^*, \mathbb{Q}_a^*\} = 0, \qquad \{\mathbb{Q}_a, \mathbb{Q}_a^*\} = \mathbb{H}_a, \tag{5.95}$$

$$\{\mathbb{Q}_b, \mathbb{Q}_b\} = 0, \qquad \{\mathbb{Q}_b^*, \mathbb{Q}_b^*\} = 0, \qquad \{\mathbb{Q}_b, \mathbb{Q}_b^*\} = \mathbb{H}_b, \tag{5.96}$$

$$\{\mathbb{Q}_a, \mathbb{Q}_b\} = 0, \qquad \{\mathbb{Q}_a^*, \mathbb{Q}_b^*\} = 0, \tag{5.97}$$

$$\{\mathbb{Q}_a, \mathbb{Q}_b^*\} = \mathbb{L}, \qquad \{\mathbb{Q}_a^*, \mathbb{Q}_b\} = \mathbb{L}^*, \tag{5.98}$$

$$[\mathbb{Q}_a, \mathbb{L}] = 0, \qquad [\mathbb{Q}_a^*, \mathbb{L}] = -(\delta - \gamma)\mathbb{Q}_b^*, \tag{5.99}$$

$$[\mathbb{Q}_a, \mathbb{L}^*] = (\delta - \gamma)\mathbb{Q}_b, \qquad [\mathbb{Q}_a^*, \mathbb{L}^*] = 0, \tag{5.100}$$

$$[\mathbb{Q}_b, \mathbb{L}] = -(\delta - \gamma)\mathbb{Q}_a, \qquad [\mathbb{Q}_b^*, \mathbb{L}] = 0, \tag{5.101}$$

$$[\mathbb{Q}_b, \mathbb{L}^*] = 0, \qquad [\mathbb{Q}_b^*, \mathbb{L}^*] = (\delta - \gamma)\mathbb{Q}_a^*, \tag{5.102}$$

$$[\mathbb{L}^*, \mathbb{L}] = -(\delta - \gamma)\mathbb{H}_a - (\delta - \gamma)\mathbb{H}_b. \tag{5.103}$$

The vector space $\mathfrak{X}$ generated by these operators is graded. To this end define the following vector spaces: $\mathfrak{X}_{\mathbb{H}} = \mathrm{span}\{\mathbb{H}_a, \mathbb{H}_b\}$, $\mathfrak{X}_a = \mathrm{span}\{\mathbb{Q}_a, \mathbb{Q}_a^*\}$, $\mathfrak{X}_b = \mathrm{span}\{\mathbb{Q}_b, \mathbb{Q}_b^*\}$, and $\mathfrak{X}_\pm = \mathrm{span}\{\mathbb{L}, \mathbb{L}^*\}$, then clearly $\mathfrak{X} = \mathfrak{X}_{\mathbb{H}} \oplus \mathfrak{X}_a \oplus \mathfrak{X}_b \oplus \mathfrak{X}_\pm$ as vector spaces.

Let $[\![\cdot, \cdot]\!]$ denote the Lie (super)bracket on $\mathfrak{X}$ where it reduces to a commutator or anti-commutator when appropriate. The above relations can be summarized neatly via the summands for $\mathfrak{X}$ in the following "Cayley table", where the "product" denotes the target space:

$[\![\cdot,\cdot]\!]$	$\mathfrak{X}_{\mathbb{H}}$	$\mathfrak{X}_a$	$\mathfrak{X}_b$	$\mathfrak{X}_\pm$
$\mathfrak{X}_{\mathbb{H}}$	$\mathfrak{X}_{\mathbb{H}}$	$\mathfrak{X}_a$	$\mathfrak{X}_b$	$\mathfrak{X}_\pm$
$\mathfrak{X}_a$	$\mathfrak{X}_a$	$\mathfrak{X}_{\mathbb{H}}$	$\mathfrak{X}_\pm$	$\mathfrak{X}_b$
$\mathfrak{X}_b$	$\mathfrak{X}_b$	$\mathfrak{X}_\pm$	$\mathfrak{X}_{\mathbb{H}}$	$\mathfrak{X}_a$
$\mathfrak{X}_\pm$	$\mathfrak{X}_\pm$	$\mathfrak{X}_b$	$\mathfrak{X}_a$	$\mathfrak{X}_{\mathbb{H}}$

The Cayley table mimics that of the usual Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and so the grading on this algebra is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. Klein graded algebras are not entirely new [51, 35, 5, 39, 40, 1]. If $(0,0)$ denotes $\mathfrak{X}_{\mathbb{H}}$, $(1,0)$ denotes $\mathfrak{X}_a$, $(0,1)$ denotes $\mathfrak{X}_b$, and $(1,1)$ denotes $\mathfrak{X}_\pm$, then the Lie (super)bracket on $\mathfrak{X}$ can be written in the following compact form:

$$[\![x,y]\!] = -(-1)^{(i_0+j_0)(i_1+j_1)}[\![y,x]\!], \tag{5.104}$$

where $x \in (i_0, j_0)$ and $y \in (i_1, j_1)$. In the case that $i_0 = 0 = j_0$ or $i_1 = 0 = j_1$, then the Lie (super)bracket is a commutator, likewise if $i_0 = 1 = j_0$ or $i_1 = 1 = j_1$, otherwise the Lie (super)bracket is an anti-commutator. This perfectly captures the above commutation and anti-commutation relations. The Jacobi identity for this $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra is

$$[\![x,[\![y,z]\!]]\!] = [\![[\![x,y]\!],z]\!] + (-1)^{(i_0+j_0)(i_1+j_1)}[\![y,[\![x,z]\!]]\!], \tag{5.105}$$

where $x \in (i_0, j_0)$ and $y \in (i_1, j_1)$.

Define now the operators $\mathcal{F}^+$ and $\mathcal{F}^-$ for fixed i and j by

$$\mathcal{F}^+ = \begin{pmatrix} \Phi_i^+ & 0 \\ 0 & \widetilde{\Phi}_i^+ \end{pmatrix}, \qquad \mathcal{F}^- = \begin{pmatrix} \Phi_j^- & 0 \\ 0 & \widetilde{\Phi}_j^- \end{pmatrix}. \tag{5.106}$$

The intertwining/skew-intertwining relations yield the following relations

$$\mathbb{Q}_a\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{Q}_a \qquad \mathbb{Q}_a^*\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{Q}_a^* \tag{5.107}$$

$$\mathbb{Q}_b\mathcal{F}^{\pm} = -\mathcal{F}^{\pm}\mathbb{Q}_b \qquad \mathbb{Q}_b^*\mathcal{F}^{\pm} = -\mathcal{F}^{\pm}\mathbb{Q}_b^* \tag{5.108}$$

$$\mathbb{L}\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{L} \qquad \mathbb{L}^*\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{L}^* \tag{5.109}$$

$$\mathbb{H}_a\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{H}_a \qquad \mathbb{H}_b\mathcal{F}^{\pm} = \mathcal{F}^{\pm}\mathbb{H}_b \tag{5.110}$$

For instance,

$$\begin{aligned}
\mathbb{Q}_a\mathcal{F}^{+} &= \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} \Phi_i^{+} & 0 \\ 0 & \widetilde{\Phi}_i^{+} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ a\Phi_i^{+} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \widetilde{\Phi}_i^{+}a & 0 \end{pmatrix} \\
&= \begin{pmatrix} \Phi_i^{+} & 0 \\ 0 & \widetilde{\Phi}_i^{+} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \\
&= \mathcal{F}^{+}\mathbb{Q}_a.
\end{aligned}$$

The other cases follow similarly.

5.8 Future Work: A Generalization of Coupled SUSY

In the theory of coupled SUSY, two ladders existed and the ladder operators alternated between the two structures. Indeed, there are only two true sectors if one generalizes the algorithm for SUSY in the following way: instead of finding a function W such that $\frac{d}{dx} + W(x)$ annihilates the lowest energy state, one finds an operator a such that a annihilates the state. In this language, it is clear that there are only two true sectors in coupled SUSY since the lowest energy state for aa^* was seen to be $\widetilde{\varphi}_{j,0}$ and it was assumed to be annihilated by the operator b^*. Developing the Hamiltonians bb^* and b^*b, one sees that the Hamiltonians that arise in this fashion are directly related to the Hamiltonians aa^* and a^*a. Thus there are two sectors which are repeated *ad infinitum* and so one may choose to view coupled SUSY as a supersymmetric theory with only two true sectors, much like the quantum mechanical harmonic oscillator having only one sector and its supersymmetric partner sectors being carbon copies thereof.

It is natural to ask what, if any, generalization of coupled SUSY there is. The most natural question to ask is if there is a supersymmetric theory with n sectors. Indeed, there is, and it may be established quite simply. We make a minor change in notation from coupled SUSY.

Definition 5.19. Let $a_1, \ldots, a_n$ be closed, densely defined operators on a Hilbert space $\mathfrak{H}$, $a_1^*, \ldots, a_n^*$ be their adjoints, and $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ with $\gamma_1 + \cdots + \gamma_n \neq 0$. Furthermore, suppose that $\operatorname{dom} a_1^* = \operatorname{dom} a_2$, $\operatorname{dom} a_2^* = \operatorname{dom} a_3$, and so on, $\operatorname{ran} a_n^* \subseteq \operatorname{dom} a_{n-1}^*$, $\operatorname{ran} a_{n-1}^* \subseteq \operatorname{dom} a_{n-2}^*$ and so on. We say that these generate an *nth order* *coupled SUSY system* if

$$a_1 a_1^* = a_2^* a_2 + \gamma_1 \tag{5.111}$$

$$a_2 a_2^* = a_3^* a_3 + \gamma_2 \tag{5.112}$$

$$\vdots \tag{5.113}$$

$$a_n a_n^* = a_1^* a_1 + \gamma_n. \tag{5.114}$$

Notice that the ordering is not the same as it is in the original realization of coupled SUSY. In coupled SUSY, $a^*a = b^*b + \gamma$ and $aa^* = bb^* + \delta$, whereas in this new notation, it would be replaced with $aa^* = b^*b + \gamma$ and $bb^* = a^*a + \delta$. This is not a significant change as it corresponds to interchanging the roles of a and a^* and replacing δ with $-\delta$. The reason for this change of notation is that it allows for the results to be very simply stated. We have the following theorem.

Theorem 5.20. *Let $a_1, \ldots, a_n$ define an nth order generalized coupled SUSY system, then the nth order operators $a_n \cdots a_2 a_1$ and $a_1^* a_2^* \cdots a_n^*$ are ladder operators for $a_1^* a_1$. Cyclic permutations of these operators act as ladder operators for the other Hamiltonians $a_j^* a_j$. The operators $a_1 a_2 \cdots a_n$ and $a_n^* \cdots a_2^* a_1^*$ are ladder operators for $a_1 a_1^*$. Cyclic permutations of these operators act as ladder operators for the other Hamiltonians $a_j a_j^*$.*

Proof. Instead of computing the commutator, we find an expression for $a_1^* a_1 a_1^* a_2^* \cdots a_n^*$ which will be closely related to the commutator.

$$a_1^* a_1 a_1^* a_2^* \cdots a_n^* = a_1^* a_1 a_1^* a_2^* \cdots a_n^*$$

$$
\begin{aligned}
&= a_1^*(a_2^*a_2 + \gamma_1)a_2^* \cdots a_n^* \\
&= \gamma_1 a_1^* a_2^* \cdots a_n^* + a_1^* a_2^* a_2 a_2^* \cdots a_n^* \\
&= \gamma_1 a_1^* a_2^* \cdots a_n^* + a_1^* a_2^*(a_3^* a_3 + \gamma_2) \cdots a_n^* \\
&= (\gamma_1 + \gamma_2) a_1^* a_2^* \cdots a_n^* + a_1^* a_2^* a_3^* a_3 \cdots a_n^* \\
&\vdots \\
&= (\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1}) a_1^* a_2^* \cdots a_n^* + a_1^* a_2^* \cdots a_n^* a_n a_n^* \\
&= (\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1}) a_1^* a_2^* \cdots a_n^* + a_1^* a_2^* \cdots a_n^*(a_1^* a_1 + \gamma_n) \\
&= (\gamma_1 + \gamma_2 + \cdots + \gamma_n) a_1^* a_2^* \cdots a_n^* + a_1^* a_2^* \cdots a_n^* a_1^* a_1
\end{aligned}
$$

Subtracting $a_1^* a_2^* \cdots a_n^* a_1^* a_1$ gives

$$[a_1^* a_1, a_1^* a_2^* \cdots a_n^*] = (\gamma_1 + \cdots + \gamma_n) a_1^* a_2^* \cdots a_n^*. \tag{5.115}$$

A similar analysis shows that $a_n \cdots a_2 a_1$ is also a ladder operator for $a_1^* a_1$. □

The operators $a_1, a_2, \ldots, a_n$ and $a_1^*, a_2^*, \ldots, a_n^*$ act as transfer operators between the different sectors, and the sectors are cyclicly related via these operators. In general, it is not clear what Lie algebra lies behind this generalization of coupled SUSY as the next result suggests.

Theorem 5.21. *The commutator of $a_n \cdots a_2 a_1$ and $a_1^* a_2^* \cdots a_n^*$ is an $(n-1)$st degree polynomial in $a_1^* a_1$.*

Proof. We have that

$$a_n \cdots a_2 a_1 a_1^* a_2^* \cdots a_n^* = a_n \cdots a_2 (a_2^* a_2 + \gamma_1) a_2^* \cdots a_n^*$$

$$= a_n \cdots (a_2 a_2^*)^2 \cdots a_n^* + \gamma_1 a_n \cdots a_2 a_2^* \cdots a_n^*$$
$$= a_n \cdots (a_3^* a_3 + \gamma_2)^2 \cdots a_n^* + \gamma_1 a_n \cdots (a_3^* a_3 + \gamma_2) \cdots a_n^*$$

A simple induction focusing on the first term yields a term of the form $(a_n a_n^*)^{n-1}$ which can be readily recognized as being related to $(a_1^* a_1)^{n-1}$ and so the claim is shown. $\square$

These results generalize those obtained in the case of traditional coupled SUSY. It is of interest to explore the Lie structure for nth order coupled SUSY and what the graded structure is.

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