TENSOR PRODUCTS OF OPERATOR SYSTEMS VIA FACTORIZATION

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Wai Hin Ng

May 2016

TENSOR PRODUCTS OF OPERATOR SYSTEMS VIA FACTORIZATION

Wai Hin Ng

Approved:

Dr. Vern Paulsen (Committee Chair)
Department of Mathematics, University of Houston

Committee Members:

Dr. David Blecher
Department of Mathematics, University of Houston

Dr. Mark Tomforde
Department of Mathematics, University of Houston

Dr. Roger Smith
Department of Mathematics, Texas A&M University

Dean, College of Natural Sciences and Mathematics
University of Houston

Acknowledgements

I would like to express my deepest gratitude to my doctoral advisor Dr. Vern I. Paulsen for his encouragement, guidance, and inspiration throughout this work, and for sharing his sense of humor while giving me a wonderful introduction to the theory of operator systems. I am indebted to him for being supportive and approachable in every stage of my research. He not only gave me the freedom to work and explore independently, but also motivated me through a series of valuable discussions. Without his enthusiasm and passion, this dissertation would not have been possible. I would also like to thank the committee members Dr. David P. Blecher, Dr. Mark Tomforde, and Dr. Roger Smith from Texas A&M University for their time to read my dissertation, and their insightful comments and suggestions.

I would also like to extend my appreciation to every member of the Department of Mathematics at the University of Houston, for making my undergraduate and graduate studies a valuable and unforgettable experience. I greatly acknowledge the support from the director of graduate study, Dr. Shanyu Ji, and the former graduate student advisor, Pamela Draughn, throughout these years. Besides, I would like to thank Dr. David P. Blecher, Dr. Mark Tomforde, and Dr. Bernhard G. Bodmann for their excellent and influential lessons during my graduate study. A special thanks to my academic brother, Dr. Ali S. Kavruk, for an insightful conversation in GPOTS 2015 at Purdue University. My sincere thanks to Dr. Michael Field and Dr. Gordon Johnson, for giving me two inspirational undergraduate years, which planted the seed of mathematics inside me. I ask for forgiveness from those members who have not been mentioned.

I owe many, many thanks to my friends for being there physically or mentally during the ups and downs of my life. Very special thanks to Mauricio, Satish, Pankaj, Angelynn, Tas, Da, Licheng, Burcin, Cihan, and Nathan for sharing the joy, the stress, the enthusiasm, and the experience throughout my graduate study. I am also indebted to my friends Yves, Perry, Nelson, and Dickson for giving me a great balance between life and academia. My heartfelt thanks to Mandy Chu, Pui Au-Yeung, Man Yui Chan, and Chun Wai Ng in Hong Kong, who are far from eyes but close to heart, for their underlying belief in me.

Finally and most importantly, this long educational journey would not have been possible without the endless support and unconditional love of my family. Thanks to my cousin Dunice, my cousin-in-law Erik, and their lovely daughter Molly, for bringing happiness to the house. I am truly thankful to my father Kwok Chuen Ng, my mother Ngan Yuk Yau, and my sister Hoi Ching Ng. I deeply appreciate their patience and care throughout the years, particularly the past year. No words can completely express my gratitude for them.

For my family, whose love is my source of strength.

TENSOR PRODUCTS AND FACTORIZATIONS OF OPERATOR SYSTEMS

An Abstract of a Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Wai Hin Ng

May 2016

Abstract

In this dissertation, we start by studying the operator system maximal tensor product, called max, in [17] from different perspectives. One approach is by the factorization technique used in Banach spaces [11] and operator spaces [5, 31]. Although in [14] it was used in establishing the operator system version of complete positive approximation property, it was not fully utilized in terms of tensor products. From this point of view, we are able to characterize max via approximate completely positive factorization through the matrix algebras.

Motivated by the significant role of self-duality in factorization, we progress to operator systems that are self-dual as matrix-ordered spaces, or in finite-dimensional case, as operator systems. We construct the self-dual operator Hilbert system SOH based on Pisier's operator Hilbert space OH [29] and prove analogous structural results of SOH. This leads us to create a tensor product of finite-dimensional operator systems via factorization through SOH, denoted by γ_{soh}. We prove various tensorial and nuclearity properties of γ_{soh}, which distinguish γ_{soh} from other known tensor products found in [8, 17, 18]. Then we extend such construction to the infinite-dimensional case and conclude that γ_{soh} indeed defines a new tensor product of operator systems.

The construction of SOH also motivates us to visit the Paulsen system $\mathcal{S}_V$ of an operator space V (see [26]). We examine some structural questions about $\mathcal{S}_V$ including the states, matrix-ordered dual, and operator system quotient. We characterize the states on $\mathcal{S}_V$, hence lead to proving that the matrix-ordered dual of $\mathcal{S}_V$ is again an operator system regardless of the operator space V. Finally, we end this dissertation with an exposition to an interesting quotient of $\mathcal{S}_V$.

Contents

Chapter 1

Background and Motivation

1.1 Introduction

The theory of tensor products is fundamental in the literature of Banach spaces and C*-algebras. In the past two decades, tensor products of operator spaces also have been studied extensively (see [2, 4, 31]). Some of the structural properties of operator spaces such as approximation property, exactness, local lifting property, and weak expectation property, are shown to be deeply related to tensor products. In recent years, a systematic study of tensor product of operator systems along with characterization of various nuclearity properties has also arisen. Through a series of papers (see [8, 9, 10, 14, 15, 17, 18]), the picture of nuclearity properties under basic algebraic constructions such as quotients, coproducts, and duality has become clear. Nevertheless, while the construction of tensor products via factorization on the categories of Banach spaces [11] and operator spaces [5, 29, 30] has been worked out, only little is known about the operator system analogue [14]. In this work, we attempt to contribute to this missing part of the big picture. More precisely, we attempt to provide construction and characterization of operator system tensor products via factorization along with various techniques using duality and quotients.

We start with an introduction to operator systems, their matrix-ordered duals, and operator system quotients. Then we give a brief survey on tensor products of operator systems. In particular, we outline some important results about the maximal, the minimal, and the commuting tensor products together with their relation to nuclearity.

In Chapter 2 we provide two characterizations of the maximal tensor product of operator systems. The Schur tensor product of operator spaces was introduced in [32]. We show that the analogous construction in the category of operator systems yields precisely the maximal tensor product. This characterization leads to a distinct, perhaps numerically more efficient, description of the maximal tensor product defined in [17]. By this characterization, we generalize a result in [32] on the tensor norm relation between the Schur tensor product and the maximal C*-norm in the category of C*-algebras.

We then proceed to another characterization of the maximal tensor product by employing the techniques of approximate completely positive factorization. More precisely, we prove that $u \in \mathcal{S} \otimes_{\max} \mathcal{T}$ is positive if and only if the associated map $\hat{u} \colon \mathcal{S}^d \to \mathcal{T}$ admits an approximate completely positive factorization through the matrix algebras M_n. We show that earlier results on (min, max)-nuclearity in [14, 15], hence the Choi-Effros-Kirchberg characterization of nuclear C*-algebras, are both direct consequences of this theorem.

In the next two chapters, we build an operator system and its associated tensor product through approximate completely positive factorization. Pisier in [29] proved that, for each dimension n, there is a unique operator space $OH(n)$ with the property that it is completely isometrically isomorphic to its dual space. In Chapter 3, we study the analogous problem in the matrix-ordered setting. Since the dual of a matrix-ordered space is still a matrix-ordered space, it is natural to ask if a matrix-ordered space is completely order-isomorphic to its dual.

Unlike the operator space case, there are many operator systems that are completely order-isomorphic to their matrix-ordered dual. Since the dual of an operator system also carries a matrix norm, it is natural to ask if an operator system is ever simultaneously completely order-isomorphic and completely norm-isomorphic to its dual. We show that this is impossible. In fact, we prove that any complete order isomorphism between an operator system and its dual has a cb-condition number that is bounded below by 2.

We look at some standard examples of self-dual and finite-dimensional operator systems and show that the corresponding cb-condition number grows unbounded as the dimension tends to infinity. We then create a "natural" operator system from $OH(n)$, that we denote by $SOH(n)$. The operator system $SOH(n)$ possesses the property that the canonical map taking a basis to its dual basis is a unital complete order isomorphism onto its dual, and it has cb-condition number of exactly 2. We then explore some further properties and applications of the operator systems $SOH(n)$. In particular, we prove that operator subsystems and quotients of $SOH(n)$ are completely order-isomorphic to $SOH(m)$ for some $m < n$.

In Chapter 4, we use "approximate cp-factorization through SOH" to construct a new tensor product of finite-dimensional operator systems, denoted by γ_{soh}, and examine some of its properties. We prove that γ_{soh} is distinct from the known ones, adding a new member to the list of tensor products introduced in [17, 18]. Moreover, we establish a few new nuclearity-related results, and then generalize such construction to infinite-dimensional operator systems.

Motivated by the construction of $SOH(n)$, we then turn to examine some natural questions about the Paulsen system by duality and operator system quotients. Paulsen [26] proved that for a complete contraction $\varphi\colon V \to \mathcal{A}$ from an operator space V into a C*-algebra $\mathcal{A}$, up to unital complete order isomorphism, there exist a unique operator system $\mathcal{S}_V$ and a unital completely positive map $\Phi\colon \mathcal{S}_V \to M_2(\mathcal{A})$ such that φ is the off-diagonal corner of Φ. Many results on completely positive maps are then extended to completely bounded maps by this theorem. Nevertheless, little is known about its matrix-ordered dual $\mathcal{S}_V^d$.

In Chapter 5, we begin the study by a characterization of the states of $\mathcal{S}_V$. We prove that for any operator space V, $\mathcal{S}_V^d$ with an appropriate order unit is an operator system. It is thus natural to study the relation between $\mathcal{S}_V^d$ and $\mathcal{S}_{V^*}$ for general operator space V. Finally, we end this work with an exposition to a natural quotient of $\mathcal{S}_V$, denoted by $\mathcal{S}_V/\mathcal{J}$. We show that $\mathcal{J}$ is completely order proximinal and deduce that $(\mathcal{S}_V/\mathcal{J})^d$ can be regarded as an operator subsystem of $\mathcal{S}_V^d$. We also prove that $\mathcal{S}_V/\mathcal{J}$, when equipped with the two operator space quotient norms obtained from either the operator space or the operator system structures of $\mathcal{S}_V$, are completely bounded-isomorphic.

1.2 Preliminaries

In this section we introduce the terminology as well as state the definitions and basic results that shall be used throughout this thesis. By a **$*$-vector space** V, we mean that V is a complex vector space equipped with an involution $*\colon V \to V$ that is conjugate linear. That is, $(v^*)^* = v$ and $(\alpha v + w)^* = \overline{\alpha} v^* + w^*$, for all $v, w \in V, \alpha \in \mathbb{C}$.

An element v is called **hermitian**, or **self-adjoint**, provided $v = v^*$. We denote V_{sa} the set of all self-adjoint elements in V. We denote $M_{m,n}(V)$ the complex vector space of $n \times m$ matrices whose entries are elements of V and denote $M_n(V)$ for $M_{n,n}(V)$. If $V = \mathbb{C}$, then we simply write $M_{m,n}$ for $M_{m,n}(\mathbb{C})$ and M_n for $M_n(\mathbb{C})$. Note that $M_n(V)$ is also a $*$-complex vector space with $[v_{ij}]^* = [v_{ji}^*]$. For $A = [a_{ij}] \in M_{m,n}$ and $X = [x_{ij}] \in M_{n,k}(V)$, by the left multiplication AX, we mean the element in $M_{m,k}(V)$ whose ij-th entry is $\sum_{r=1}^n a_{ir}x_{rj}$, for $1 \leq i \leq m, 1 \leq j \leq k$. We define the right multiplication in a similar way.

Let V be a $*$-complex vector space. A **matrix order**, **matricial order**, or **matricial cones**, on V is a family $\{C_n \colon C_n \subset M_n(V)\}_{n=1}^\infty$ satisfying the following axioms:

1. $C_n \cap (-C_n) = \{0\}$.
2. $M_n(V)$ is the complex span of C_n.
3. For each $n, m \in \mathbb{N}$, if $A = [a_{ij}] \in M_{n,m}$ and $[v_{ij}] \in C_n$, then $AVA^* = [\sum_{k,l} a_{ik}v_{kl}\overline{a_{lj}}] \in C_m$.

The third axiom is called **compatibility**. The pair $(V, \{C_n\})$ is called a **matrix-ordered $*$-vector space** or simply **matrix-ordered space**. An element in C_n is called a **positive** element in $M_n(V)$ and we sometimes denote $M_n(V)^+$ for C_n and V^+ for C_1. For each $n \in \mathbb{N}$, there is a natural order structure on $M_n(V)$ induced by C_n given by $A \leq B$ if and only if $B - A \in C_n$.

In a matrix-ordered space, when we only consider the ground level (V, V^+), an element $e \in V_{sa}$ is called an **order unit** for V if for every $x \in V_{sa}$, there exists a positive real number r such that $re + x \geq 0$. We call e an **Archimedean order unit** if e is an order unit and satisfies the following property: For any $x \in V$, if $re + x \geq 0$ for all $r > 0$, then $x \geq 0$. We call the triple (V, V^+, e) an **Archimedean order space with unit** (AOU space).

In the case of matrix level $(V, \{C_n\})$, we say that e is a **matrix order unit** for $(V, \{C_n\})$ if the corresponding element

$$e_n = \begin{bmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{bmatrix}$$

is an order unit for $M_n(V)$, for every $n \in \mathbb{N}$. We call e an **Archimedean matrix order unit**, provided e_n is an Archimedean order unit for each n.

A triple $(V, \{C_n\}, e)$, where $(V, \{C_n\})$ is a matrix-ordered space with an Archimedean matrix order unit e, is called an **(abstract) operator system**. In brief we usually call e "unit" and use $\mathcal{S}$ (or $\mathcal{T}$, $\mathcal{R}$) to denote the operator system $(V, \{C_n\}, e)$. We often use e or 1 for the unit of $\mathcal{S}$, and add subscripts when there are two or more operator systems. We also write $\mathcal{S}^+ = C_1$ and $M_n(\mathcal{S})^+ = C_n$.

An important example of matrix-ordered spaces is when we take V to be a $*$-closed subspace of $B(\mathcal{H})$, the C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. We equip V with the induced matricial cone structure. More precisely, we identify $M_n(B(\mathcal{H}))$ with $B(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$, the C^*-algebra of bounded linear operators on the direct sum of n copies of $\mathcal{H}$. Via this identification, we regard $M_n(V)$ again as a $*$-closed subspace of $M_n(B(\mathcal{H}))$ with $C_n = M_n(V) \cap M_n(B(\mathcal{H}))^+$, where $M_n(B(\mathcal{H}))^+$ denotes the cone of positive elements of $M_n(B(\mathcal{H}))$. One can check that $(V, \{M_n(V)\})$ is a matrix-ordered space. By a **concrete operator system**, we mean a unital self-adjoint subspace V of $B(\mathcal{H})$, or in general a C*-algebra by the Gelfand-Naimark theorem. Note that the matrix-ordered space $(V, \{M_n(V)\})$ together with the identity operator I on $\mathcal{H}$ satisfies the axioms of an abstract operator system. Therefore, every concrete operator system is an abstract operator system. For the converse, in the next paragraph we introduce the appropriate morphisms of operator systems.

Suppose $\mathcal{S}$ and $\mathcal{T}$ are abstract operator systems and $\varphi\colon \mathcal{S} \to \mathcal{T}$ is a linear map. We call φ a **unital map** if $\varphi(1_{\mathcal{S}}) = 1_{\mathcal{T}}$. It is called **positive** provided $\varphi(x) \geq 0$ for every $x \in \mathcal{S}^+$; or equivalently, $\varphi(\mathcal{S}^+) \subset \mathcal{T}^+$. We say that φ is **completely positive** if its n-th amplification $\varphi^{(n)}\colon M_n(\mathcal{S}) \to M_n(\mathcal{T})$ given by $[x_{ij}] \mapsto [\varphi(x_{ij})]$ is positive for each $n \in \mathbb{N}$; or equivalently, $\varphi^{(n)}(M_n(\mathcal{S})^+) \subset M_n(\mathcal{T})^+$ for every n. We call φ a **complete order embedding** if it is an injective completely positive map with the property that whenever $[\varphi(x_{ij})]$ is positive in $M_n(\mathcal{T})$, then $[x_{ij}] \in M_n(\mathcal{S})$ is positive. A bijective map $\varphi\colon \mathcal{S} \to \mathcal{T}$ is called a **complete order isomorphism** if both φ and φ^{-1} are complete order embeddings. If φ is unital, then we say $\mathcal{S}$ is **unitally completely order-isomorphic** to $\mathcal{T}$ and denote it by $\mathcal{S} \cong_{ucoi} \mathcal{T}$. A unital and self-adjoint subspace $\mathcal{T}$ of an operator system $\mathcal{S}$ is again an operator system together with the induced matricial order structure; we call $\mathcal{T}$ an **operator subsystem** of $\mathcal{S}$. The inclusion map $\mathcal{T} \hookrightarrow \mathcal{S}$ is a unital complete order embedding, and sometimes we denote it by $T \subset_{ucoi} \mathcal{S}$. We write $\mathcal{O}$ for the category whose objects are operator systems and morphisms are completely positive maps; and $\mathcal{O}_1$ is the category whose objects are operator systems and morphisms are unital completely positive maps.

We now state the abstract characterization theorem of operator systems, due to Choi and Effros [6]:

Theorem 1.1. *Up to a unital complete order isomorphism, all the abstract and con-*

crete operator systems coincide. That is, if $\mathcal{S}$ is an operator system, then there is a Hilbert space $\mathcal{H}$ and a unital $*$-linear map $\varphi\colon \mathcal{S} \to B(\mathcal{H})$ which is a complete order embedding.

Finally we remark that every operator system $\mathcal{S}$ has a canonical operator space structure. A subspace X of some $B(\mathcal{H})$ or a C*-algebra is called a **concrete** operator space. We refer the reader to [26] for an introduction to this subject along with the abstract characterization by Ruan. If $\mathcal{S}$ is an operator system, then any concrete representation $\varphi\colon \mathcal{S} \to B(\mathcal{H})$ endows $\mathcal{S}$ with an operator space structure. It turns out that this structure is independent of the representation and is intrinsic in the matricial order structure. More precisely, the family of operator space norms on $\mathcal{S}$ is given as follows: For $X = [x_{ij}] \in M_n(\mathcal{S})$

$$||X||_n = \inf \left\{ r > 0\colon re_{2n} + \begin{bmatrix} 0 & [x_{ij}] \\ [x_{ji}^*] & 0 \end{bmatrix} \in M_{2n}(\mathcal{S})^+ \right\}.$$

This is known as the **canonical operator space structure** of $\mathcal{S}$. In the later chapters, we will look at this structure on $\mathcal{S}$ and use results in operator space theory. We refer the reader to [31] for an excellent resource on this subject.

1.2.1 Duality of operator systems

We assume that the reader is familiar with the basic definitions and properties of operator spaces, operator systems, completely bounded and completely positive maps. In this subsection, we define dual spaces of operator spaces and operator systems, and outline some important aspects of the latter. For more details, the reader should see [10] and the books [26, 30].

If V is an operator space, then the space of bounded linear functionals on V, denoted by V^d, comes equipped with a natural **dual matrix-norm**. Briefly, a matrix of linear functionals $F = [f_{i,j}] \in M_n(V^d)$ is identified with a linear map $F\colon V \to M_n$ and we set $\|(f_{i,j})\|_n = \|F\|_{cb}$.

Given a matrix-ordered space V and a $*$-closed subspace $V_1 \subset V$, note that if $V_1 \subseteq V$ is a $*$-invariant vector subspace, then the cones $C_n \cap M_n(V_1)$ endow V_1 with a matrix-order that we call the **subspace order,** or more simply, we refer to $V_1 \subseteq V$ as the **matrix-ordered subspace.** Given two matrix-ordered spaces V and W we call a map $\phi\colon V \to W$ **completely positive** provided that $\phi^{(n)}\colon M_n(V) \to M_n(W)$ is positive for all n.

Given a matrix-ordered space V, we let $V^\ddagger$ denote the vector space of all linear functionals on V. Given a linear functional $f\colon V \to \mathbb{C}$, if we let $f^*\colon V \to \mathbb{C}$ be the linear functional $f^*(v) = \overline{f(v^*)}$, then this makes $V^\ddagger$ a $*$-vector space. We identify an $n \times n$ matrix of linear functionals $[f_{ij}]$ with the linear map $F\colon V \to M_n$ defined by $F(v) = [f_{ij}(v)]$, and set $M_n(V^\ddagger)^+$ equal to the cone of completely positive maps. Then this gives a compatible family of proper cones on the dual on $V^\ddagger$, but in general $M_n(V^\ddagger)^+$ does not span $M_n(V^\ddagger)$. When V is also a normed space, then we let V^d denote the space of bounded linear functionals on V, which is a subspace of $V^\ddagger$ and is endowed with the subspace order.

However, when V is an operator system, then V^d endowed with this set of cones is a matrix-ordered space and we refer to this as the **matrix-ordered dual of V.** The easiest way to see that these cones span, is to use Wittstock's decomposition theorem [26, 34] which says that the completely bounded maps on an operator system are the complex span of the completely positive maps.

Given two matrix-order spaces V and W and a linear map $\phi\colon V \to W$, the **dual**

map of ϕ, is the map $\phi^d\colon W^d \to V^d$ given by $\phi^d(f)(x) = f(\phi(x))$. With the definitions above, it is evident that ϕ is completely positive if and only if ϕ^d is completely positive. The reader should note that when V and W are operator systems, the result still holds except that V^d and W^d are considered only as matrix-ordered spaces since in general they are not operator systems. However, when V and W are finite-dimensional, their dual spaces are operator systems due to the following theorem by Choi and Effros [6].

Theorem 1.2. *If $\mathcal{S}$ is a finite-dimensional operator system, then there exist faithful states on $\mathcal{S}$ and each faithful state is an Archimedean matrix order unit for $\mathcal{S}^d$. Therefore, $\mathcal{S}^d$ is an operator system.*

Henceforth, for finite-dimensional operator system $\mathcal{S}$, we regard $\mathcal{S}^d$ as an operator system dual with a fixed faithful state.

1.2.2 Quotients of Operator Systems

In this subsection we give an brief exposition to the quotient theory developed in [18]. For more details, we refer the reader to [15, 18, 27, 28]. A $*$-closed subspace $\mathcal{J}$ of an operator system $\mathcal{S}$ is called a **kernel** if $\mathcal{J}$ is a kernel of a unital completely positive map $\phi\colon \mathcal{S} \to \mathcal{T}$, for some operator system $\mathcal{T}$. Here is a brief characterization of kernels from Proposition 3.1 in [18]which we shall use in the sequel.

Proposition 1.3. Let $\mathcal{J}$ be a subspace of an operator system $\mathcal{S}$. Then the following are equivalent:

1. $\mathcal{J}$ is a kernel of $\mathcal{S}$.
2. There exists a completely positive map $\phi\colon \mathcal{S} \to \mathcal{T}$ such that $J = \ker \phi$.

3. There exists a collection $\{f_i\}_{i\in I}$ of states of $\mathcal{S}$ such that $\mathcal{J} = \bigcap_{i\in I} \ker f_i$.

We remark that the first two statements justify that kernels in $\mathcal{O}$ and $\mathcal{O}_1$ are all equivalent. Also, the last statement provides an intrinsic description of kernels that only relies on the states of $\mathcal{S}$.

Suppose $\mathcal{J}$ is a kernel of $\mathcal{S}$ and $q\colon \mathcal{S} \to \mathcal{S}/\mathcal{J}$ is its natural quotient map onto the algebraic quotient. The space $\mathcal{S}/\mathcal{J}$ is a $*$-vector space with involution $(x+\mathcal{J})^* = x^* + \mathcal{J}$. Define $D_n = D_n(\mathcal{S}/\mathcal{J}) = q^{(n)}(M_n(\mathcal{S}^+))$; that is,

$$D_n = \{[x_{ij} + \mathcal{J}] \in M_n(\mathcal{S}/\mathcal{J})\colon \text{ there exists } y_{ij} \in \mathcal{J} \text{ so that } [x_{ij} + y_{ij}] \in M_n(\mathcal{S})^+\}.$$

For each $n \in \mathbb{N}$, D_n defines a proper cone in $M_n(\mathcal{S}/\mathcal{J})_{sa}$, and $\{D_n\}$ is a compatible family. Equipped with these $\{D_n\}$, $\mathcal{S}/\mathcal{J}$ becomes a matrix-ordered space with matrix order unit $e+\mathcal{J}$. Nevertheless, often $e+\mathcal{J}$ is not Archimedean, so $(\mathcal{S}/\mathcal{J}, \{D_n\}, e+\mathcal{J})$ is not an operator system.

For this matter, in [27, 28] it is shown that every matrix-ordered order space $(Q, \{D_n\}, 1)$, where 1 is a non-Archimedean matrix order unit, gives rise to an operator system through the **Archimedeanization process**. In brief, through this process we expand each D_n to a slightly larger cone C_n, by taking its closure with respect to an order topology generated by seminorms. Although this process is rather technical (see Section 3 in [28]), it turns out that the structure of $\{C_n\}$ is fairly natural.

Proposition 1.4. Let $\mathcal{S}$ be an operator system and $\mathcal{J} \subset \mathcal{S}$ be a kernel. If we define

$$C_n(\mathcal{S}/\mathcal{J}) = \{[x_{ij} + \mathcal{J}] \in M_n(\mathcal{S}/\mathcal{J})\colon \forall \varepsilon > 0, \varepsilon(e+\mathcal{J})_n + [x_{ij}] \in D_n\},$$

then $(\mathcal{S}/\mathcal{J}, \{C_n\}, e + \mathcal{J})$ is a matrix-ordered space with Archimedean matrix order unit, hence an operator system. Moreover, the quotient map $q\colon \mathcal{S} \to \mathcal{S}/\mathcal{J}$ is completely positive.

Definition 1.5. By an **operator system quotient** $\mathcal{S}/\mathcal{J}$, we always refer to this triple defined in Proposition 1.4. In the case when $C_1 = D_1$, we say that $\mathcal{J}$ is **order proximinal**, and it is called **complete order proximinal** if $C_n = D_n$ for all $n \in \mathbb{N}$.

We end the subsection with the following characterization of operator system quotients in terms of a universal property, similar to quotient objects in other categories.

Proposition 1.6. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\mathcal{J}$ be a kernel in $\mathcal{S}$. If $\phi\colon \mathcal{S} \to \mathcal{T}$ is a unital completely positive map with $\mathcal{J} \subset \ker(\phi)$, then the induced map $\tilde{\phi}\colon \mathcal{S}/\mathcal{J} \to \mathcal{T}$ given by $\tilde{\phi}(x + \mathcal{J}) = \phi(x)$ is unital completely positive. Moreover if $\mathcal{R}$ is an operator system and $\psi\colon \mathcal{S} \to \mathcal{R}$ is unital completely positive, with the property that whenever $\phi\colon \mathcal{S} \to \mathcal{T}$ is completely positive with $\mathcal{J} \subset \ker(\phi)$, there exists a unique unital completely positive map $\hat{\phi}\colon \mathcal{R} \to \mathcal{T}$ such that $\hat{\phi} \circ \psi = \phi$; then there exists a completely order isomorphism $\gamma\colon \mathcal{R} \to \mathcal{S}/\mathcal{J}$ such that $\gamma \circ \psi = q$.

1.2.3 Tensor Products of Operator Systems

We outline a few basic facts about tensor products of operator systems. We also give a brief survey on the maximal, the minimal, and the commuting tensor products, as well as their relations to some nuclearity results. We refer the reader to [10, 15, 17, 18] for the details.

Definition 1.7. Given a pair of operator systems $(\mathcal{S}, \{P_n\}_{n=1}^{\infty}, 1_{\mathcal{S}})$ and $(\mathcal{T}, \{Q_n\}_{n=1}^{\infty}, 1_{\mathcal{T}})$, by an **operator system structure** on $\mathcal{S} \otimes \mathcal{T}$, we mean a family $\tau = \{C_n\}_{n=1}^{\infty}$ of cones, where $C_n \subset M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying:

(T1) $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty}, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ is an operator system denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$.

(T2) $P_n \otimes Q_m \in C_{nm}$, for all $n, m \in \mathbb{N}$.

(T3) If $\phi\colon \mathcal{S} \to M_n$ and $\psi\colon \mathcal{T} \to M_m$ are unital completely positive maps, then $\phi \otimes \psi\colon \mathcal{S} \otimes_\tau \mathcal{T} \to M_{nm}$ is a unital completely positive map.

By an **operator system tensor product**, we mean a mapping $\tau\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, $\tau(\mathcal{S}, \mathcal{T}) = \mathcal{S} \otimes_\tau \mathcal{T}$, that satisfies axioms (T1) to (T3). We say τ is **functorial**, provided in addition it satisfies the following property:

(T4) Given operator systems $\mathcal{S}_i$ and $\mathcal{T}_i$, $i = 1, 2$, if $\phi_i\colon \mathcal{S}_i \to \mathcal{T}_i$ is unital completely positive, then $\phi_1 \otimes \phi_2\colon \mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \to \mathcal{T}_1 \otimes_\tau \mathcal{T}_2$ is unital completely positive.

Let τ be an operator system tensor product. If for all operator systems $\mathcal{S}$ and $\mathcal{T}$, the map $\theta\colon x \otimes y \mapsto y \otimes x$ is a unital complete order isomorphism from $\mathcal{S} \otimes_\tau \mathcal{T}$ onto $\mathcal{T} \otimes_\tau \mathcal{S}$, then τ is called **symmetric**. If for any three operator systems $\mathcal{S}$, $\mathcal{T}$, $\mathcal{R}$, the natural map from $(\mathcal{S} \otimes_\tau \mathcal{T}) \otimes_\tau \mathcal{R}$ to $\mathcal{S} \otimes_\tau (\mathcal{T} \otimes_\tau \mathcal{R})$ is a unital complete order isomorphism, then τ is called **associative**.

We say that τ is **left injective**, provided whenever $\phi\colon \mathcal{S} \to \mathcal{R}$ is a complete order embedding, then for any operator system $\mathcal{T}$, the map $\phi \otimes id$ is a complete order embedding from $\mathcal{S} \otimes_\tau \mathcal{T}$ into $\mathcal{R} \otimes_\tau \mathcal{T}$. It is equivalent to require that for every $n \in \mathbb{N}$, $(\phi \otimes id)^{(n)}$ is bijective between $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ and $M_n(\mathcal{R} \otimes_\tau \mathcal{T})^+$. Right injectivity is defined in a similar vein, and we say τ is **injective** if it is both left and right injective.

We say that τ is left **projective**, provided whenever $q\colon \mathcal{S} \to \mathcal{R}$ is a complete quotient map, then for any operator system $\mathcal{T}$, the map $q \otimes id$ is a complete quotient from $\mathcal{S} \otimes_\tau \mathcal{T}$ onto $\mathcal{R} \otimes_\tau \mathcal{T}$. It is equivalent to require that for every $n \in \mathbb{N}$, every $u \in M_n(\mathcal{R} \otimes_\tau \mathcal{T})^+$, and every $\varepsilon > 0$, there is $\tilde{u_\varepsilon} \in M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ so that $q \otimes id(\tilde{u_\varepsilon}) =$

$u + \varepsilon(I_n \otimes 1_{\mathcal{R}} \otimes 1_{\mathcal{T}})$. Right projectivity is defined similarly and we say τ is **projective** if it is both left and right projective.

Given two operator system tensor products τ_1 and τ_2, we say that τ_1 is **greater than** τ_2 provided the identity map from $\mathcal{S} \otimes_{\tau_1} \mathcal{T}$ to $\mathcal{S} \otimes_{\tau_2} \mathcal{T}$ is completely positive; equivalently, $M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subset M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$, for every n. We denote it by $\tau_2 \leq \tau_1$. With this ordering, we have a lattice structure on the family of operator system tensor products introduced in [17, Proposition 7.1]:

Proposition 1.8. The family of operator system tensor products with respect to $\leq$ defined above forms a complete lattice. The family of functorial operator system tensor products is a complete sublattice of this lattice.

In recent years, various results are established using nuclearity and operator system tensor products (see [8, 9, 15, 17, 18]). We devote the rest to this chapter on three important tensor products and some of their nuclearity results.

The Maximal Tensor Product

Given operator systems $\mathcal{S}$ and $\mathcal{T}$, their maximal tensor product, denoted by max, is equipped the smallest family of cones for which the algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ forms an operator system satisfying axioms (T1) to (T3). In $\mathcal{O}$, the maximal tensor product is the natural analogue of the projective tensor norm of operator spaces, as well as a generalization of the maximal C*-norm on C*-algebras. More precisely, if X and Y are operator spaces, and $\mathcal{S}_X$ and $\mathcal{S}_Y$ are their Paulsen systems, respectively, then the projective operator space tensor product $X \hat{\otimes} Y$ can be embedded completely isometrically into $\mathcal{S}_X \otimes_{\max} \mathcal{S}_Y$. Also, when $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras, the operator system $\mathcal{A} \otimes_{\max} \mathcal{B}$ is unitally completely order-isomorphic to the C*-maximal tensor product $\mathcal{A} \otimes_{\mathrm{C^*\text{-}max}} \mathcal{B}$ before completion. For more details, we refer the reader to Section 5 of [17].

The construction of the maximal tensor product is as follows. Given operator systems $\mathcal{S}$ and $\mathcal{T}$, we first define the family of cones

$$\mathcal{D}_n^{\max} = \mathcal{D}_n^{\max}(\mathcal{S}, \mathcal{T}) = \{A(P \otimes Q)A^* \colon P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+,$$
$$A \in M_{n,km}, k, m \in \mathbb{N}\}.$$

We shall remark the following useful representation of $\mathcal{D}_1^{\max}$.

Lemma 1.9. *Every $u \in \mathcal{D}_1^{\max}$ can be represented as $u = \sum p_{ij} \otimes q_{ij}$ for some $[p_{ij}] \in M_n(\mathcal{S})^+$ and $[q_{ij}] \in M_n(\mathcal{T})^+$.*

Proof. If $u = A(P \otimes Q)A^*$ as above with $A \in M_{1,km}$, note that u is then the sum of the entries of the Kronecker tensor product $(A \begin{bmatrix} P & \dots & P \\ \vdots & & \vdots \\ P & \dots & P \end{bmatrix} A^*) \otimes Q$, where the operator matrix is in $M_m(M_k(\mathcal{S}))^+$. Since we can replace Q by $\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$ of an appropriate size and likewise for the first operator matrix, we deduce such representation as claimed. $\square$

This matricial order $\{\mathcal{D}_n^{\max}\}$ is then a compatible family with matrix order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$. However, it is not Archimedean, so we complete the cones through the Archimedeanization process as described in the previous section by defining the following cones

$$\mathcal{C}_n^{\max}(\mathcal{S}, \mathcal{T}) = \{U \in M_n(\mathcal{S} \otimes \mathcal{T}) \colon r(1_{\mathcal{S}} \otimes 1_{\mathcal{T}}) + U \in \mathcal{D}_n(\mathcal{S}, \mathcal{T}), \forall r > 0\}.$$

Likewise we denote $\mathcal{C}_n^{\max}(\mathcal{S}, \mathcal{T}) = \mathcal{C}_n^{\max}$. Now $\mathcal{S} \otimes \mathcal{T}$ equipped with this family $\{\mathcal{C}_n^{\max}\}_{n=1}^{\infty}$ satisfies axioms (T1) to (T4), and it defines a symmetric and associative operator system structure. We call it the **maximal tensor product** of $\mathcal{S}$ and $\mathcal{T}$ and denote it $\mathcal{S} \otimes_{\max} \mathcal{T}$. It turns out that the maximal tensor product is projective by Proposition 1.6 in [10].

With respect to the lattice structure of operator system tensor products, the maximal tensor product is the largest one. Also, it has the following universal property by Theorem 5.8 in [17].

Theorem 1.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A bilinear map $\phi\colon \mathcal{S} \times \mathcal{T} \to \mathcal{B}(\mathcal{H})$ is jointly completely positive if and only if its linearization $L_\phi\colon \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(\mathcal{H})$ is a completely positive map. Moreover, if τ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ satisfying this property, then $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\max} \mathcal{T}$.*

Corollary 1.11. If we take $\mathcal{B}(\mathcal{H}) = \mathbb{C}$ in the above theorem, we obtain another important aspect of the maximal tensor product:

$$(\mathcal{S} \otimes_{\max} \mathcal{T})^{d,+} = CP(\mathcal{S}, \mathcal{T}^d),$$

where the $CP(\mathcal{S}, \mathcal{T})$ denotes the cone of all completely positive maps from $\mathcal{S}$ to $\mathcal{T}$.

This statement is precisely the operator system analogue of a result by Lance in [21]. The following lemma is in [17] and will be used in the next chapter. We include the proof for completeness.

Lemma 1.12. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\{C_n\}_{n=1}^\infty$ be a compatible family of cones of $\mathcal{S} \otimes \mathcal{T}$ satisfying axiom (T2). Then $\mathcal{D}_n^{\max} \subset C_n$.*

Proof. If $P \in M_n(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$, then axiom (T2) implies $P \otimes Q \in C_{nm}$. By compatibility of $\{C_n\}$, $A(P \otimes Q)A^* \in C_k$, for all $A \in M_{k,nm}$; hence $\mathcal{D}_n^{\max} \subset C_n$. $\square$

In [17], it is shown that for operator spaces V and W, the projective tensor product $V \stackrel{\wedge}{\otimes} W$ can be completely isometrically embedded onto the (1,2) entry of the maximal tensor product of their corresponding Paulsen systems; that is, $V \stackrel{\wedge}{\otimes} W \subset \mathcal{S}_V \otimes_{\max} \mathcal{S}_W$, complete norm isometrically. Also, any unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are as well operator systems; in the same paper it is proved that their C*-maximal tensor product $\mathcal{A} \otimes_{\text{C*-max}} \mathcal{B}$ is completely order-isomorphic to $\mathcal{A} \otimes_{\max} \mathcal{B}$. Consequently, the maximal tensor product in $\mathcal{O}$ is an analogue of the projective tensor norm on operator spaces and maximal C*-norm on C*-algebras.

The Minimal Tensor Product

In contrast to the maximal tensor product, the minimal tensor product, denoted by min, of two operator systems $\mathcal{S}$ and $\mathcal{T}$ is equipped the largest family of cones for which $\mathcal{S} \otimes \mathcal{T}$ forms an operator system satisfying axioms (T1) to (T3). In $\mathcal{O}$, the minimal tensor product is the natural analogue of the injective tensor norm of operator spaces, as well as a generalization of the minimal C*-norm on C*-algebras. We refer the reader to Section 4 of [17] for the analogous statements about the relation among the injective operator space tensor product, the minimal tensor product, and the C*-minimal tensor product.

The construction of $\mathcal{S} \otimes_{\min} \mathcal{T}$ is rather natural. Given operator systems $\mathcal{S}$ and $\mathcal{T}$, we define the family of cones

$$\begin{aligned} \mathcal{C}_n^{\min} = \mathcal{C}_n^{\min}(\mathcal{S}, \mathcal{T}) = \{[x_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T})\colon\ & [(\phi \otimes \psi)(x_{ij})] \in M_{nkm}^+, \\ & \text{for all unital completely positive} \\ & \phi\colon \mathcal{S} \to M_k, \psi\colon \mathcal{T} \to M_m, \forall k, m \in \mathbb{N}\}. \end{aligned}$$

The triple $(\mathcal{S}\otimes\mathcal{T}, \{\mathcal{C}_n^{\min}\}, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ satisfies axioms (T1) to (T4), and it defines a symmetric and associative operator system structure. We call it the **minimal tensor product** of $\mathcal{S}$ and $\mathcal{T}$ and denote it $\mathcal{S}\otimes_{\max}\mathcal{T}$. It turns out that the minimal tensor product is injective, due to the following characterization.

Theorem 1.13. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems, and let $\iota_{\mathcal{S}}\colon \mathcal{S}\to B(\mathcal{H})$ and $\iota_{\mathcal{T}}\colon \mathcal{T}\to B(\mathcal{K})$ be unital complete order embeddings. Then $\mathcal{S}\otimes_{\min}\mathcal{T}$ is the operator system structure on $\mathcal{S}\otimes\mathcal{T}$ arising from the embedding $\iota_{\mathcal{S}}\otimes\iota_{\mathcal{T}}\colon \mathcal{S}\otimes\mathcal{T}\to B(\mathcal{H}\otimes\mathcal{K})$.*

Therefore, the minimal tensor product is the spatial tensor product in $\mathcal{O}$. Moreover, with respect to the lattice structure of operator system tensor products, the minimal tensor product is the smallest one.

In [17], given operator spaces V and W, the injective tensor product $V\overset{\vee}{\otimes}W$ can be completely isometrically embedded onto the (1,2)-entry of the minimal tensor product of their corresponding Paulsen systems; that is, $V\overset{\vee}{\otimes}W\subset\mathcal{S}_V\otimes_{\min}\mathcal{S}_W$, complete norm isometrically. Moreover, for any unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, the C*-minimal tensor product $\mathcal{A}\otimes_{\text{C*-min}}\mathcal{B}$ is completely order-isomorphic to $\mathcal{A}\otimes_{\min}\mathcal{B}$. Consequently, the minimal tensor product in $\mathcal{O}$ is an analogue of the injective tensor norm on operator spaces and minimal C*-norm on C*-algebras.

Another important aspect of the minimal tensor product arises when we combine the duality result by Farenick and Paulsen [10, Proposition 1.9] and Corollary 1.11.

Theorem 1.14. *Suppose $\mathcal{S}$ and $\mathcal{T}$ are finite-dimensional operator systems, then $\mathcal{S}^d\otimes_{\max}\mathcal{T}^d$ and $(\mathcal{S}\otimes_{\min}\mathcal{T})^d$ are complete order-isomorphic. Moreover,*

$$(\mathcal{S}\otimes_{\min}\mathcal{T})^+ = (\mathcal{S}^d\otimes_{\max}\mathcal{T}^d)^{d,+} = CP(\mathcal{S},\mathcal{T}^d) = CP(\mathcal{S}^d,\mathcal{T}).$$

In Chapter 2, we will see a similar description of $(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ using approximate completely positive factorization.

The Commuting Tensor Product

The commuting (or maximal commuting) tensor product, denoted by c, is another important operator system tensor product. It coincides with the maximal C*-tensor norm on the category of unital C*-algebra, but it is different from the maximal tensor product on $\mathcal{O}$. The construction arises by using completely positive maps with commuting ranges. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We define the family of cones

$$\begin{aligned} \mathcal{C}_n^{comm} = \mathcal{C}_n^{comm}(\mathcal{S}, \mathcal{T}) = \{[x_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T})\colon [(\phi \otimes \psi)(x_{ij})] \geq 0 \\ \text{for all unital completely positive} \\ \phi\colon \mathcal{S} \to B(\mathcal{H}), \psi\colon \mathcal{T} \to B(\mathcal{H}) \text{ that commute}\}. \end{aligned}$$

The triple $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^{comm}\}, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ satisfies axioms (T1) to (T3), and the resultant operator system is denoted by $\mathcal{S} \otimes_c \mathcal{T}$. The commuting tensor product is functorial and symmetric. The following are important relations to the maximal tensor product from Theorem 6.4 in [17] and Proposition 3.4 in [15]. This proposition uses the idea of the universal C*-algebra generated by an operator system introduced in [20].

Theorem 1.15. *If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{S}$ is an operator system, then $\mathcal{A} \otimes_c \mathcal{S} = \mathcal{A} \otimes_{\max} \mathcal{S}$.*

Proposition 1.16. Suppose $\mathcal{S}$ and $\mathcal{T}$ are operator systems. Then $\mathcal{S} \otimes_c \mathcal{T} \subset C_u^*(\mathcal{S}) \otimes_{\max} \mathcal{T}$, where $C_u^*(\mathcal{S})$ denotes the the universal C*-algebra of $\mathcal{S}$.

Nuclearity

In Chapter 4, we will deal with various operator system tensor products and nuclearity. Here we only include a small amount of information regarding the topic. For the details, we refer the reader to [15, 17, 18].

Definition 1.17. Let $\alpha \leq \beta$ be operator system tensor products and $\mathcal{S}$ be an operator system. We say that $\mathcal{S}$ is $\boldsymbol{(\alpha, \beta)}$**-nuclear** if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$.

In [14], Han and Paulsen characterized $(\min, \max)$-nuclearity, which is part of the motivation of our work in Chapter 2. We first introduce the definition of completely positive factorization property (CPFP). An operator system $\mathcal{S}$ is said to have **CPFP** if there exist nets of unital complete positive maps $\phi_\lambda \colon \mathcal{S} \to M_{n_\lambda}$ and $\psi_\lambda \colon M_{n_\lambda} \to \mathcal{S}$, such that $\psi_\lambda \circ \phi_\lambda$ converges to the identity map in the point-norm topology; that is, for every $x \in \mathcal{S}$, $||(\psi_\lambda \circ \phi_\lambda)x - x|| \to 0$. The following is Corollary 3.2 in [14].

Theorem 1.18. *Let $\mathcal{S}$ be an operator system. Then $\mathcal{S}$ is* $(\min, \max)$*-nuclear if and only if $\mathcal{S}$ has CPFP.*

Since unital C*-algebras are operator systems, this theorem generalizes the Choi-Effros-Kirchberg characterization of nuclear C*-algebras in [7, 19]. Consequently, the classical term "nuclearity" coincides with $(\min, \max)$-nuclearity. We also state the following nuclearity results concerning on min, c, and max from [15] that will be used in the later chapters.

Proposition 1.19. An operator system $\mathcal{S}$ is $(\min, c)$-nuclear if and only if $\mathcal{S} \otimes_{\min} \mathcal{A} = \mathcal{S} \otimes_{\max} \mathcal{A}$, for every unital C*-algebra $\mathcal{A}$.

Proposition 1.20. The following are equivalent for a finite-dimensional operator system $\mathcal{S}$:

1. $\mathcal{S}$ is (c, max)-nuclear.

2. $\mathcal{S}$ is unitally completely order-isomorphic to a C*-algebra.

3. $\mathcal{S} \otimes_c \mathcal{S}^d = \mathcal{S} \otimes_{\max} \mathcal{S}^d$.

Note: Material in this dissertation has appeared elsewhere. Chapter 2 is available in [22]; and Chapters 3 and 4 will be published in [23].

Chapter 2

The Maximal Tensor Product

In this chapter we provide two characterizations of the maximal tensor product of operator systems. In the first section, we introduce the Schur tensor product of operator spaces studied by Rajpal, Kumar, and Itoh in [32]. We show that the analogous construction in the category of operator systems yields precisely the maximal tensor product. This characterization leads to a distinct, perhaps numerically more efficient, description of the maximal tensor product. By this characterization, we generalize a result in [32] on the tensor norm relation between the Schur tensor product and the maximal C*-norm in the category of C*-algebras.

In the second section, we characterize the maximal tensor product by employing the techniques of approximate completely positive factorization. More precisely, we prove that $u \in \mathcal{S} \otimes_{\max} \mathcal{T}$ is positive if and only if the associated map $\hat{u} \colon \mathcal{S}^d \to \mathcal{T}$ admits an approximate completely positive factorization through the matrix algebras M_n. We show that earlier results on (min, max)-nuclearity in [14, 15], or Theorem 1.18, are direct consequences of this theorem. Therefore, it as well generalizes the Choi-Effros-Kirchberg characterization of nuclear C*-algebras [7, 19]. It is worthwhile to note that such technique in constructing operator system tensor products is the analogue of that in operator spaces and Banach spaces. We will visit this construction again in Chapter 4.

2.1 The Schur Tensor Product

Definition 2.1. Given matrix-ordered spaces V and W, $X = [x_{ij}] \in M_n(V)^+$, and $Y = [y_{ij}] \in M_n(W)^+$, we define the **Schur tensor product** $X \circ Y$ to be

$$X \circ Y = [x_{ij} \otimes y_{ij}] \in M_n(V \otimes W).$$

If V and W are operator spaces, define the Schur tensor norm for each $U \in M_n(V \otimes W)$ to be

$$||U||_s = \inf\{||A||||X||||Y||||B||\},$$

where $U = A(X \circ Y)B$, for some $A \in M_{n,m}$, $B \in M_{m,n}$, $X \in M_m(V)$, and $Y \in M_m(W)$. In [32], it is shown that $|| \cdot ||_s$ is matrix norm and the completion of $V \otimes W$ in this norm is an operator space. It is called the Schur tensor product of V and W, denoted by $V \otimes^s W$. Moreover, it is distinct from the operator space projective tensor product.

We are interested in the analogous construction in the category of operator systems. Firstly, the following lemmas outline an interesting relation between the Schur tensor product and the ordinary algebraic tensor product.

Lemma 2.2. *Every $X \circ Y \in M_n(\mathcal{S} \otimes \mathcal{T})$ can be regarded as $\mathcal{E}(X \otimes Y)\mathcal{E}^*$, for some $\mathcal{E} \in M_{n,n^2}$.*

Proof. Let $\{E_{ij}\}_{i,j=1}^n$ denote the standard matrix units of M_n and regard $X \otimes Y$ as the Kronecker tensor product. In the case when $n = 2$, note that

$$\begin{bmatrix} E_{11} & E_{22} \end{bmatrix} X \otimes Y \begin{bmatrix} E_{11} & E_{22} \end{bmatrix}^* =$$

$$\begin{bmatrix} E_{11} & E_{22} \end{bmatrix} \begin{bmatrix} x_{11} \otimes y_{11} & x_{11} \otimes y_{12} & x_{12} \otimes y_{11} & x_{12} \otimes y_{12} \\ x_{11} \otimes y_{21} & x_{11} \otimes y_{22} & x_{12} \otimes y_{21} & x_{12} \otimes y_{22} \\ x_{21} \otimes y_{11} & x_{21} \otimes y_{12} & x_{22} \otimes y_{11} & x_{22} \otimes y_{12} \\ x_{21} \otimes y_{21} & x_{21} \otimes y_{22} & x_{22} \otimes y_{21} & x_{22} \otimes y_{22} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} \otimes y_{11} & x_{12} \otimes y_{12} \\ x_{21} \otimes y_{21} & x_{22} \otimes y_{22} \end{bmatrix} = X \circ Y.$$

In general, a similar calculation shows that $X \circ Y = \mathcal{E}(X \otimes Y)\mathcal{E}^*$, where $\mathcal{E} = [E_{11}\ E_{22}\ \ldots\ E_{nn}]$. □

Lemma 2.3. *Every $P \in M_n(\mathcal{S} \otimes \mathcal{T})$ can be written as $P = A(X \circ Y)B$, for some $X \in M_k(\mathcal{S})$, $Y \in M_k(\mathcal{T})$, $A \in M_{n,k}$ and $B \in M_{k,n}$. In particular, we may take $B = A^*$.*

Proof. Write P as a sum of matrices whose entries are elementary tensors, that is, $P = \sum_{l=1}^m U^l$, where $U^l = [x_{ij}^l \otimes y_{ij}^l] \in M_n(\mathcal{S} \otimes \mathcal{T})$. Let $U = U^1 \oplus \cdots \oplus U^m$, so

$$U = \begin{bmatrix} U^1 & 0 & \ldots & 0 \\ 0 & U^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & U^m \end{bmatrix}$$

which is $X \circ Y$ for some $X \in M_{nm}(\mathcal{S})$ and $Y \in M_{nm}(\mathcal{T})$. Now let $A = [I_n\ I_n\ \ldots\ I_n] \in$

$M_{n,nm}$ with m copies of I_n. Then, it is easy to see that $AUA^* = \sum_{l=1}^{m} U^l = P$. $\square$

Hence, $X \circ Y$ and $X \otimes Y$ are almost identical except by a $*$-conjugation of matrices. Motivated by this observation and the construction of the maximal tensor product, we define the following family of cones.

Definition 2.4. Given operator systems $\mathcal{S}$ and $\mathcal{T}$, we define

$$\begin{aligned}\mathcal{C}_n^s(\mathcal{S} \otimes \mathcal{T}) = \{&A(X \circ Y)A^* \in M_n(\mathcal{S} \otimes \mathcal{T}) \colon \\ &X \in M_k(\mathcal{S})^+, Y \in M_k(\mathcal{T})^+, A \in M_{n,k}, k \in \mathbb{N}\}.\end{aligned}$$

In brief we denote $\mathcal{C}_n^s(\mathcal{S} \otimes \mathcal{T}) = \mathcal{C}_n^s$.

Proposition 2.5. The family $\{\mathcal{C}_n^s\}$ defines a matrix order on $\mathcal{S} \otimes \mathcal{T}$ with matrix order unit $1 \otimes 1$.

Proof. We first check that $\mathcal{C}_n^s$ is a cone of $M_n(\mathcal{S} \otimes \mathcal{T})$. It is obvious from definition that $\mathcal{C}_n^s \subset M_n(\mathcal{S} \otimes \mathcal{T})_{sa}$. Let $A(X_1 \circ Y_1)A^*$ and $B(X_2 \circ Y_2)B^*$ be in $\mathcal{C}_n^s$, where $X_1 \in M_k(\mathcal{S})$, $Y_1 \in M_k(\mathcal{T})$, $X_2 \in M_m(\mathcal{S})$, $Y_2 \in M_m(\mathcal{T})$, $A \in M_{n,k}(\mathbb{C})$, and $B \in M_{n,m}(\mathbb{C})$. Let

$$X = X_1 \oplus X_2 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \in M_{k+m}(\mathcal{S})^+,$$

and likewise $Y = Y_1 \oplus Y_2 \in M_{k+m}(\mathcal{T})^+$. Consider $[A\ B] \in M_{n,k+m}$, then

$$\begin{aligned}\begin{bmatrix} A & B \end{bmatrix} (X \circ Y) \begin{bmatrix} A & B \end{bmatrix}^* &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_1 \circ Y_1 & 0 \\ 0 & X_2 \circ Y_2 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix}^* \\ &= A(X_1 \circ Y_1)A^* + B(X_2 \circ Y_2)B^*\end{aligned}$$

is in $\mathcal{C}_n^s$. If $t > 0$, then $t(A(X \circ Y)A^*) = (\sqrt{t}A)(X \circ Y)(\sqrt{t}A)^* \in \mathcal{C}_n^s$. Also, if $B \in M_{r,n}$ then $(BA)(X \circ Y)(BA)^* \in \mathcal{C}_r^s$. Therefore, $\{\mathcal{C}_n^s\}_{n=1}^\infty$ is a compatible family of cones on $\mathcal{S} \otimes \mathcal{T}$.

Finally, to see that they are proper, we claim that in fact $\mathcal{C}_n^s \subset \mathcal{D}_n^{\max}$. Indeed, let $A(X \circ Y)A^* \in \mathcal{C}_n^s$, for some $X \in M_k(\mathcal{S})^+$, $Y \in M_k(\mathcal{T})^+$, and $A \in M_{n,k}$. Then by Lemma 2.2,

$$A(X \circ Y)A^* = A(\mathcal{E}(X \otimes Y)\mathcal{E}^*)A^* = (A\mathcal{E})(X \otimes Y)(A\mathcal{E})^*,$$

which is in $\mathcal{D}_n^{\max}$ by definition. Since the latter cone is proper, $-\mathcal{C}_n^s \cap \mathcal{C}_n^s = \{0\}$. The fact that $1 \otimes 1$ is a matrix order unit with respect to $\{\mathcal{C}_n^s\}$ follows from the inclusion $\mathcal{C}_n^s \subset \mathcal{D}_n^{\max}$ and that $1 \otimes 1$ is a matrix order unit with respect to $\mathcal{D}_n^{\max}$. Consequently, $\{\mathcal{C}_n^s\}$ defines a matrix order on $\mathcal{S} \otimes \mathcal{T}$. $\square$

From the last paragraph of the proof, we see that $\mathcal{C}_n^s \subset \mathcal{D}_n^{\max}$. In fact, one can further deduce that $\mathcal{C}_n^s = \mathcal{D}_n^{\max}$ after proving that this family satisfies axiom (T2).

Lemma 2.6. *The family $\{\mathcal{C}_n^s\}_{n=1}^\infty$ satisfies axiom (T2). That is, given $X \in M_n(\mathcal{S})^+$ and $Y \in M_m(\mathcal{T})^+$, $X \otimes Y \in \mathcal{C}_{nm}^s$.*

Proof. Let X and Y be as above, note that we may view

$$\begin{aligned} X \otimes Y &= [x_{ij} \otimes Y]_{i,j=1}^n \\ &= \begin{bmatrix} x_{11} \otimes J_m & \dots & x_{n1} \otimes J_m \\ \vdots & \ddots & \vdots \\ x_{n1} \otimes J_m & \dots & x_{nn} \otimes J_m \end{bmatrix}_{nm \times nm} \circ \begin{bmatrix} Y & \dots & Y \\ \vdots & \ddots & \vdots \\ Y & \dots & Y \end{bmatrix}_{nm \times nm}, \end{aligned}$$

where $J_k \in M_k$ is the matrix whose entries are all 1. It is easy to see that the second matrix in the above equation is $Y \otimes J_n$. A straight-forward calculation shows that for each $k \in \mathbb{N}$, J_k has eigenvalues 0 and k, so $Y \otimes J_n \in M_{nm}(\mathcal{T})^+$. On the other hand, the first matrix is unitarily equivalent to $X \otimes J_m$, which is also positive in $M_{nm}(\mathcal{S})$. Therefore, $X \otimes Y = (X \otimes J_m) \circ (Y \otimes J_n) \in \mathcal{C}^s_{nm}$ and the family $\{\mathcal{C}^s_n\}_{n=1}^{\infty}$ satisfies axiom (T2). $\square$

Now by Lemma 1.12, we have the reverse inclusion $\mathcal{D}^{\max}_n \subset \mathcal{C}^s_n$, so the two families of cones coincide. In particular, Lemma 1.2 follows easily: every $u \in \mathcal{D}^{\max}_1 = \mathcal{C}^s_1$ can be represented as $u = A(P \circ Q)A^* = (A^*PA) \circ Q$, for some $A \in M_{1,n}$, $P \in M_n(\mathcal{S})^+$, and $Q \in M_n(\mathcal{T})^+$. If we archimedeanize the cones $\{\mathcal{C}^s_n\}_{n=1}^{\infty}$, then we obtain the Schur tensor product of operator systems and denote it $\mathcal{S} \otimes_s \mathcal{T}$; and it is unitally completely order isomorhpic to $\mathcal{S} \otimes_{\max} \mathcal{T}$.

Theorem 2.7. *The cones $\mathcal{C}^s_n = \mathcal{D}^{\max}_n$, for every $n \in \mathbb{N}$. Consequently, for operator systems, the Schur tensor product is the maximal tensor product, i.e. $\mathcal{S} \otimes_s \mathcal{T} = \mathcal{S} \otimes_{\max} \mathcal{T}$.*

Given operator systems $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \otimes_{\max} \mathcal{T}$ possesses a canonical operator space matrix norm $|| \cdot ||_{\text{osy-max}}$; that is, given $U \in M_n(\mathcal{S} \otimes_{\max} \mathcal{T})$,

$$||U||_{\text{osy-max}} = \inf \left\{ r \colon \begin{bmatrix} rI & U \\ U^* & rI \end{bmatrix} \in M_{2n}(\mathcal{S} \otimes_{\max} \mathcal{T})^+ \right\}.$$

In particular, since $\mathcal{A} \otimes_{\text{C*-max}} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ for unital C*-algebras, the C*-maximal tensor norm $|| \cdot ||_{\text{C*-max}}$ is precisely $|| \cdot ||_{\text{osy-max}}$ for unital C*-algebras. The following proposition is a generalized version of $|| \cdot ||_{\text{C*-max}} \leq || \cdot ||_s$ in [32].

Proposition 2.8. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Then the identity map $\phi\colon \mathcal{S} \otimes^s \mathcal{T} \to \mathcal{S} \otimes_{\max} \mathcal{T}$ is a complete contraction between the two operator spaces.

Proof. Let $||U||_s < 1$, then by scaling, there exist scalar contractions A, B and $X \in M_n(S)$ and $Y \in M_n(T)$, $||X||, ||Y|| \leq 1$ such that $U = A(X \circ Y)B$. Hence, the matrices $P = \left[\begin{smallmatrix} I & X \\ X^* & I \end{smallmatrix}\right] \in M_{2n}(S)^+$ and $Q = \left[\begin{smallmatrix} I & Y \\ Y^* & I \end{smallmatrix}\right] \in M_{2n}(T)^+$. Note that

$$\begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} P \circ Q \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} AA^* & A(X \circ Y)B \\ B^*(X^* \circ Y^*)A^* & B^*B \end{bmatrix} = \begin{bmatrix} AA^* & U \\ U^* & B^*B \end{bmatrix},$$

which is in $M_{2n}(\mathcal{S} \otimes_s \mathcal{T})^+ = M_{2n}(\mathcal{S} \otimes_{\max} \mathcal{T})^+$.

On the other hand, since A and B are scalar contractions, $I - AA^*$ and $I - B^*B$ are positive in M_n. Thus, the operator matrix $\left[\begin{smallmatrix} I-AA^* & 0 \\ 0 & I-B^*B \end{smallmatrix}\right]$ is positive in $M_{2n}(\mathcal{S} \otimes_{\max} \mathcal{T})$. By adding the two matrices, we obtain $\left[\begin{smallmatrix} I & U \\ U^* & I \end{smallmatrix}\right] \in M_{2n}(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ which implies that $||U||_{\text{osy-max}} \leq 1$. □

2.2 Factorization Through the Matrix Algebras

We now turn to study the maximal tensor product using factorization. Recall that every $u = \sum_{i=1}^n x_i \otimes y_i \in \mathcal{S} \otimes \mathcal{T}$ may be regarded as a linear map $\hat{u}\colon \mathcal{S}^d \to \mathcal{T}$, $\hat{u}(f) = \sum_{i=1}^n f(x_i) y_i$, where $\mathcal{S}^d$ is the linear dual of $\mathcal{S}$. The map $\hat{u}$ is independent of representation of u and $u \mapsto \hat{u}$ is an one-to-one correspondence between $\mathcal{S} \otimes \mathcal{T}$ and $B(\mathcal{S}^d, \mathcal{T})$, where the latter is the space of bounded linear maps from the linear dual $\mathcal{S}^d$ to $\mathcal{T}$.

In this section, we use the duality results from [10]. For the rest of the chapter, to ensure $\mathcal{S}^d$ is an operator system, we assume $\mathcal{S}$ and $\mathcal{T}$ to be finite-dimensional.

Fix a basis $\{y_1 = 1_{\mathcal{T}}, \ldots, y_m\}$ for $\mathcal{T}$, where $y_i = y_i^*$ and $||y_i|| = 1$, so that every $u \in \mathcal{S} \otimes \mathcal{T}$ has a unique representation $u = \sum_{i=1}^m x_i \otimes y_i$, for some $x_i \in \mathcal{S}$. To obtain the main result in this chapter, we introduce a temporary norm on $\mathcal{S} \otimes \mathcal{T}$ by setting $|||u||| = \sum_{i=1}^m ||x_i||$.

Lemma 2.9. *If $u = \sum_{i=1}^m x_i \otimes y_i \in \mathcal{S} \otimes \mathcal{T}$, where $x_i = x_i^*$, then $|||u|||(1_{\mathcal{S}} \otimes 1_{\mathcal{T}}) + u \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$.*

Proof. Because

$$\begin{bmatrix} ||s_i||1 & s_i \\ s_i & ||s_i||1 \end{bmatrix} \in M_2(\mathcal{S})^+, \qquad \begin{bmatrix} 1 & t_i \\ t_i & 1 \end{bmatrix} \in M_2(\mathcal{T})^+,$$

when we form their Schur tensor product, we obtain

$$\begin{bmatrix} ||s_i||1 \otimes 1 & s_i \otimes t_i \\ s_i \otimes t_i & ||s_i||1 \otimes 1 \end{bmatrix} \in \mathcal{C}_2^s = \mathcal{D}_2^{\max}.$$

By $*$-conjugating this matrix by $[1, 1]$, we deduce that $||s_i||(1 \otimes 1) + s_i \otimes t_i \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$ for each i, thus the sum $|||u|||(1 \otimes 1) + u \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$. □

Lemma 2.10. *Let u_λ be a net in $\mathcal{S} \otimes \mathcal{T}$. Then $|||u_\lambda||| \to 0$ in $\mathcal{S} \otimes \mathcal{T}$ if and only if for each $f \in \mathcal{S}^d$, $||\hat{u_\lambda}(f)||_{\mathcal{T}} \to 0$.*

Proof. Write each $u_\lambda = \sum_{i=1}^m x_i^\lambda \otimes y_i$, then $|||u_\lambda||| \to 0$ implies that $\lim_\lambda ||x_i^\lambda|| \to 0$ for each $i \in \{1, \ldots m\}$, which is equivalent to require that $(x_i^\lambda) \to 0$ in the weak topology. Thus for each $f \in \mathcal{S}^d$,

$$||\hat{u_\lambda}(f)||_{\mathcal{T}} \leq \sum_{i=1}^m |f(x_i^\lambda)| \cdot ||y_i||_{\mathcal{T}} \to 0.$$

Conversely, it suffices to show that for each $i \in \{1, \dots, m\}$, $\lim_\lambda ||x_i^\lambda|| = 0$. Note that for $t = \sum_{i=1}^m c_i y_i \in \mathcal{T}$, $\alpha(t) := \sum_{i=1}^m |c_i|$ defines a norm on $\mathcal{T}$. Since $\mathcal{T}$ is finite-dimensional, $||t||_{\mathcal{T}} \leq \alpha(t) \leq K||t||_{\mathcal{T}}$ for some $K > 0$. For each $f \in \mathcal{S}^d$, taking $c_i = f(x_i^\lambda)$ shows that

$$\sum_{i=1}^m |f(x_i^\lambda)| \leq K||\hat{u}_\lambda(f)||_{\mathcal{T}} \to 0.$$

Hence for each $f \in \mathcal{S}^d$ and $i \in \{1, \dots, n\}$, $|f(x_i^\lambda)| \to 0$. The latter condition is equivalent to $(x_i^\lambda) \to 0$ in the weak topology, which coincides with the norm topology because $\mathcal{S}$ is finite-dimensional. $\square$

Definition 2.11. We say that a linear map $\theta\colon \mathcal{S} \to \mathcal{T}$ admits an **approximate completely positive factorization through $\boldsymbol{M_n}$**, provided there exists nets of completely positive maps $\phi_\lambda\colon \mathcal{S} \to M_{n_\lambda}$ and $\psi_\lambda\colon M_{n_\lambda} \to \mathcal{T}$ such that $\psi_\lambda \circ \phi_\lambda$ converges to θ in the point-norm topology. In brief we say that θ factors through M_n approximately. An operator system $\mathcal{S}$ is said to have complete positive approximation property (CPAP) if the identity map factors through M_n approximately.

In [14] it is shown that $\mathcal{S}$ is $(\min, \max)$-nuclear if and only if $\mathcal{S}$ has CPAP. We now establish the main theorem in the section.

Theorem 2.12. *Let $\mathcal{S}$ and $\mathcal{T}$ be finite-dimensional operator systems and $u \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$. The following are equivalent:*

1. *u is positive in $\mathcal{S} \otimes_{\max} \mathcal{T}$.*

2. The map $\hat{u}\colon \mathcal{S}^d \to \mathcal{T}$ factors through M_n approximately:

$$\begin{tikzcd} \mathcal{S}^d \arrow[rr, "\hat{u}"] \arrow[dr, "\varphi_\lambda"'] & & \mathcal{T} \\ & M_{n_\lambda} \arrow[ur, "\psi_\lambda"'] & \end{tikzcd}$$

Proof. Suppose $u \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$. Then for each $\varepsilon > 0$, $u_\varepsilon = \varepsilon(1 \otimes 1) + u$ is in $\mathcal{D}_1^{max}(\mathcal{S}, \mathcal{T})$. By Lemma 1.4, it can be written as $u_\varepsilon = \sum p_{ij}^\varepsilon \otimes q_{ij}^\varepsilon$, where $P_\varepsilon = [p_{ij}^\varepsilon] \in M_{n_\varepsilon}(\mathcal{S})^+$ and $Q_\varepsilon = [q_{ij}^\varepsilon] \in M_{n_\varepsilon}(\mathcal{T})^+$. Define $\varphi_\varepsilon\colon \mathcal{S}^d \to M_{n_\varepsilon}$ by $\varphi_\varepsilon(f) = [f(p_{ij}^\varepsilon)]$ and $\psi_\varepsilon\colon M_{n_\varepsilon} \to \mathcal{T}$ by $\psi_\varepsilon([a_{ij}]) = \sum_{i,j} a_{ij} q_{ij}^\varepsilon$. Note that φ_ε is completely positive by definition of $\mathcal{S}^d$. For ψ_ε, first consider the completely positive map $[a_{ij}] \mapsto [a_{ij}] \otimes Q_\varepsilon$. We then regard $[a_{ij}] \otimes Q_\varepsilon$ as the matrix $[q_{ij}^\varepsilon [a_{kl}]]_{i,j}^{n_\varepsilon}$ and $*$-conjugate it by $[E_{11}\ E_{12}\ \ldots\ E_{1n_\varepsilon}]$ to obtain the matrix $[q_{ij}^\varepsilon a_{ij}]_{i,j=1}^{n_\varepsilon} \in M_{n_\varepsilon}(\mathcal{T})^+$. Now $*$-conjugate it by the row vector of length n_ε whose entries are 1; this yields $\sum_{i,j} a_{ij} q_{ij}^\varepsilon$, and ψ_ε is completely positive. It follows that $\hat{u_\varepsilon} = \psi_\varepsilon \circ \varphi_\varepsilon$ and it converges to $\hat{u}$ as $\varepsilon \to 0$ in the point-norm topology.

Conversely, every $\psi_\lambda \circ \varphi_\lambda$ corresponds to a $w_\lambda \in \mathcal{S} \otimes \mathcal{T}$ so that $\hat{w_\lambda} = \psi_\lambda \circ \varphi_\lambda$. By the identification $CP(\mathcal{S}^d, M_n) = \mathcal{S} \otimes_{\min} M_n$ from Theorem 1.14, together with the facts that $M_n(\mathcal{S})^+ = \mathcal{S} \otimes_{\min} M_n$ and $M_n = M_n^d$, we can identify φ_λ to $P_\lambda = [p_{ij}^\lambda] \in M_{n_\lambda}(\mathcal{S})^+$. Similarly, we identify ψ_λ to $Q_\lambda = [q_{ij}^\lambda] \in M_{n_\lambda}(\mathcal{T})^+$, which shows that $w_\lambda = \sum_{i,j}^{n_\lambda} p_{ij}^\lambda \otimes q_{ij}^\lambda \in \mathcal{D}_1^{\max}(\mathcal{S} \otimes \mathcal{T})$. By the point-norm convergence and the last lemma, $\lim_\lambda |||u - w_\lambda||| \to 0$. Now for each λ, take $\varepsilon_\lambda = |||u - w_\lambda|||$, and Lemma 2.9 asserts that $\varepsilon_\lambda(1 \otimes 1) + (u - w_\lambda) \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$. For each $\varepsilon > 0$ there exists a λ, such that $\varepsilon_\lambda < \varepsilon$ and $\varepsilon(1 \otimes 1) + (u - w_\lambda) \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$. Hence $\varepsilon 1 \otimes 1 + u \in \mathcal{D}_1^{\max}(\mathcal{S}, \mathcal{T})$ and $u \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$. $\square$

By the identification $M_m(\mathcal{S} \otimes_{\max} \mathcal{T}) \cong \mathcal{S} \otimes_{\max} M_m(\mathcal{T})$, we establish the following characterization of the matricial order structure of the maximal tensor product.

Theorem 2.13. *An element $U \in M_m(\mathcal{S} \otimes_{\max} \mathcal{T})$ is positive if and only if $\hat{U}\colon \mathcal{S}^d \to M_m(\mathcal{T})$ factors through M_n approximately.*

We would like to remark that this result is rather interesting. In [10] or Theorem 1.14, we have $\mathcal{S} \otimes_{\min} \mathcal{T} = (\mathcal{S}^d \otimes_{\max} \mathcal{T}^d)^d$. Combining with the result after Theorem 1.10, we deduce that $(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = CP(\mathcal{S}^d, \mathcal{T})$; whereas by Theorem 2.12, $(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ corresponds to a proper subcone of $CP(\mathcal{S}^d, \mathcal{T})$ whose elements factor through M_n approximately. Since the minimal and maximal tensor products each represents respectively the largest and smallest matricial cone structure one can equip on $\mathcal{S} \otimes \mathcal{T}$, it brings up the natural question about the corresponding subsets of the those cones with respect to other tensor products in [17].

Here we show that symmetry and projectivity of the maximal tensor product can also be obtained by this diagram.

Proposition 2.14. The maximal tensor product is symmetric and projective.

Proof. Let $u = \sum s_i \otimes t_i \in \mathcal{S} \otimes_{\max} \mathcal{T}$. By dualizing the diagram in Theorem 2.12, one sees that

$$\begin{tikzcd} \mathcal{T}^d \arrow[rr, "(\hat{u})^d"] \arrow[dr, "\psi_\lambda^d"'] & & \mathcal{S}^{dd} = \mathcal{S} \\ & M_{n_\lambda}^d = M_{n_\lambda} \arrow[ur, "\varphi_\lambda^d"'] & \end{tikzcd}$$

where $(\hat{u})^d$ is the map $g \mapsto \sum g(t_i)s_i$. Consequently, $\sum t_i \otimes s_i \in (\mathcal{T} \otimes_{\max} \mathcal{S})^+$ if and only if the above diagram holds, which by duality is equivalent to Theorem 2.12 (2). This shows that $\mathcal{S} \otimes_{\max} \mathcal{T} \cong_{ucoi} \mathcal{T} \otimes_{\max} \mathcal{S}$ at the ground level. At each matrix level

n, identifying $M_n(\mathcal{S} \otimes_{\max} \mathcal{T}) = \mathcal{S} \otimes_{\max} M_n(\mathcal{T})$ and replacing $\mathcal{T}$ by $M_n(\mathcal{T})$ proves symmetry of the maximal tensor product.

For projectivity, first consider a complete quotient map $q \colon \mathcal{S} \to \mathcal{R}$. We claim that every $u \in (\mathcal{R} \otimes_{\max} \mathcal{T})^+$ can be lifted to some $w \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$. Indeed, by Theorem 2.12 there are φ_λ and ψ_λ such that $\psi_\lambda \circ \varphi_\lambda$ converges to u in the point-norm topology. Since $q^d \colon \mathcal{R}^d \to \mathcal{S}^d$ is a complete order inclusion, by the Arveson's extension theorem [1, 26], there is a completely positive $\Phi_\lambda \colon \mathcal{S}^d \to M_{n_\lambda}$ extending φ_λ. Hence, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{R}^d \arrow[rr, "\hat{u}"] \arrow[d, "q^d"'] \arrow[dr, "\varphi_\lambda"] & & \mathcal{T} \\
\mathcal{S}^d \arrow[r, dashed, "\Phi_\lambda"'] & M_{n_\lambda} \arrow[ur, "\psi_\lambda"'] &
\end{tikzcd}$$

Let $[s_{ij}^\lambda] \in M_{n_\lambda}(\mathcal{S})^+$ be the corresponding matrix of Φ_λ and likewise for $[t_{ij}^\lambda] \in M_{n_\lambda}(\mathcal{T})^+$ of ψ_λ. Then $w_\lambda = \sum_{i,j} s_{ij}^\lambda \otimes t_{ij}^\lambda \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$ by the Schur characterization and $\hat{w} = \psi_\lambda \circ \Phi_\lambda$. To this end, we claim that there is a subnet w_{λ_α} converging to some positive w such that $\hat{w} \circ q^d = \hat{u}$.

Let δ_0 denote the unit in $\mathcal{R}^d \subset_{coi} \mathcal{S}^d$. Then $||\hat{w_\lambda}(\delta_0)|| = ||\psi_\lambda \circ \varphi_\lambda(\delta_0)|| \to ||\hat{u}(\delta_0)||$ asserts there is λ_0 such that the set $\{||\hat{w_\lambda}(\delta_0)|| \colon \lambda > \lambda_0\}$ is bounded. However, for completely positive maps, $||\hat{w_\lambda}(\delta_0)|| = ||\hat{w_\lambda}||_{cb} = ||\hat{w_\lambda}||$ and the latter norm also defines a norm on $\mathcal{S} \otimes \mathcal{T}$. By the equivalence of norm topologies, $\{w_\lambda \colon \lambda > \lambda_0\}$ is bounded in $(\mathcal{S} \otimes_{\max} \mathcal{T})^+$ and possesses a convergent subnet $w_{\lambda_\alpha} \to w \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$. Therefore for each $f \in \mathcal{R}^d$,

$$||(\hat{w} \circ q^d - \hat{u})f|| = ||(\lim_\alpha \hat{w_{\lambda_\alpha}} \circ q^d - \hat{u})f|| = ||(\lim_\alpha \psi_{\lambda_\alpha} \circ \Phi_{\lambda_\alpha} \circ q^d - \hat{u})f||$$

$$= ||(\lim_{\alpha} \psi_{\lambda_\alpha} \circ \varphi_{\lambda_\alpha} - \hat{u})f|| = \lim_{\alpha} ||(\psi_{\lambda_\alpha} \circ \varphi_{\lambda_\alpha} - \hat{u})f||$$
$$= \lim_{\lambda} ||(\psi_\lambda \circ \varphi_\lambda - \hat{u})f|| \to 0,$$

where the second line follows from Lemma 2.10. Consequently every positive $u \in \mathcal{R} \otimes_{\max} \mathcal{T}$ can be lifted to a positive $w \in \mathcal{S} \otimes_{\max} \mathcal{T}$. This implies that for every such u and for each $\varepsilon > 0$, the element $w + \varepsilon(1_{\mathcal{S}} \otimes 1_{\mathcal{T}}) \in (\mathcal{S} \otimes_{\max} \mathcal{T})^+$ satisfies $(q \otimes id)(w + \varepsilon(1_{\mathcal{S}} \otimes 1_{\mathcal{T}})) = u + \varepsilon(1_{\mathcal{R}} \otimes 1_{\mathcal{T}})$.

Finally, again by identifying $M_n(\mathcal{R} \otimes_{\max} \mathcal{T})$ to $\mathcal{R} \otimes_{\max} M_n(\mathcal{T})$ and likewise for $\mathcal{S} \otimes_{\max} M_n(\mathcal{T})$, we prove that the maximal tensor product is left projective. By symmetry, it is right projective and hence projective. $\square$

At last, we prove that this characterization of the maximal tensor product indeed leads to the $(\min, \max)$-nuclearity result in [14, 15].

Corollary 2.15. Let $\mathcal{T}$ be a finite-dimensional operator system with basis $\{y_i : 1 \leq i \leq m\}$ and let δ_i be the dual basis of y_i for $\mathcal{T}^d$. Then $u = \sum_{i=1}^m \delta_i \otimes y_i \in \mathcal{T}^d \otimes_{\max} \mathcal{T}$ is positive if and only if $\mathcal{T}$ is (min, max)-nuclear.

Proof. Let $\mathcal{S} = \mathcal{T}^d$ and note that $\hat{u}$ is the identity map on $\mathcal{T}$. Moreover, $u \in (\mathcal{T}^d \otimes_{\max} \mathcal{T})^+$ if and only if $\hat{u}$ factors through M_n approximately, which by Theorem 1.18, if and only if $\mathcal{T}$ is (min, max)-nuclear. $\square$

At last we remark that although Theorem 2.12 characterizes max in the finite-dimensional case, there is a natural extension to the infinite-dimensional case. We will cover this in a more general setting in Section 4.2.

Chapter 3

The Operator Hilbert System SOH

In [29], Pisier constructed, for each cardinal n, a unique operator space $OH(n)$ that is completely isometrically isomorphic to its operator space dual $OH(n)^*$. In this chapter, we study the analogous problem with operator systems. Unlike the operator space case, there are many operator systems that are completely order-isomorphic to their matrix-ordered dual. Since an operator system is also an operator space, its dual comes equipped with two structures: an operator space structure and a matrix-ordered structure. It is natural to ask if an operator system is ever simultaneously completely order-isomorphic and completely norm-isomorphic to its dual. We show that the answer is negative. Indeed, we prove that the cb-condition number of any complete order isomorphism between an operator system and its dual is bounded below by 2.

Next we proceed to create a natural operator system from $OH(n)$, that we denote by $SOH(n)$. The operator system $SOH(n)$ possesses the property that the canonical map taking a basis to its dual basis is a unital complete order isomorphism onto its dual, and it has cb-condition number of exactly 2. We then explore some further properties and applications of the operator systems $SOH(n)$. In particular, we prove that operator subsystems and quotients of $SOH(n)$ are completely order-isomorphic to $SOH(m)$ for some $m < n$.

3.1 Operator System and Operator Space Duality

We begin with some examples. We always identify the dual of $\mathbb{C}^n$ with $\mathbb{C}^n$ again via the map that sends the standard basis $\{e_j\}$ to the dual basis $\{\delta_j\}$.

Example 3.1. The identification of ℓ_n^∞ with the continuous functions on an n point space makes ℓ_n^∞ into an operator system with $\sum_j A_j \otimes e_j \in M_m(\ell_n^\infty)^+$ if and only if $A_j \in M_m^+$ for all j. Moreover, a map $\Phi : \ell_n^\infty \to M_m$ with $\Phi(e_j) = A_j$ is completely positive if and only if $A_j \in M_m^+$ for all j. From this it follows that the map $e_j \to \delta_j$ is a complete order isomorphism between ℓ_n^∞ and $(\ell_n^\infty)^d$. Thus, as a matrix-ordered space ℓ_n^∞ is self-dual.

On the other hand ℓ_n^∞ is also an operator space and the normed dual is ℓ_n^1 via the same identification. The operator space structure on $(\ell_n^\infty)^d$ is the operator space $MAX(\ell_n^1) = \text{span}\{u_1, ..., u_n\} \subseteq C^*(\mathbb{F}_n)$ where $C^*(\mathbb{F}_n)$ denotes the full C*-algebra of the free group on n generators and u_j are the generators [35]. In this case the norm and cb-norm of the identity map $id : \ell_n^\infty \to \ell_n^1$ is n. The cb-condition number is $\|id\|_{cb}\|id^{-1}\|_{cb} = n$.

Example 3.2. If we consider M_n as an operator system with the usual structure, then [28] the map that sends the matrix units $E_{i,j}$ to their dual basis $\{\delta_{i,j}\}$ defines a complete order isomorphism between M_n and M_n^d. This map sends the identity operator $I_n = \sum_{j=1}^n E_{j,j}$ to the trace functional tr, where $tr([a_{i,j}]) = \sum_{j=1}^n a_{j,j}$. Thus,

M_n is also completely order-isomorphic to its dual.

However, recall that the normed dual, with this same identification, is the trace class matrices S_n^1, together with their operator space structure. Again the norm, cb-norm, and cb-condition number of the identity map (between these n^2-dimensional spaces) is n.

Thus, in both these examples we have operator systems that are completely order-isomorphic to their ordered duals, but the identification does not preserve the operator space structure of the dual.

3.2 The Operator Hilbert System SOH

In this section, for each cardinal number n, we introduce an operator system $SOH(n)$ of dimension $n+1$ based on Pisier's self-dual operator space $OH(n)$ and analyze their properties. In particular, we prove that these operator systems are self-dual as matrix-ordered spaces and that the natural map from $\phi : SOH(n) \to SOH(n)^d$ satisifes $\|\phi\|_{cb} \cdot \|\phi^{-1}\|_{cb} = 2$, which we show is as close to being a complete isometry as is possible for any operator system that is completely order-isomorphic to its dual.

We begin with a result that shows that the lower bound of 2 is sharp.

Proposition 3.3. Let $\mathcal{S}$ be an operator system of dimension at least 2 and assume that $\phi : \mathcal{S} \to \mathcal{S}^d$ is a complete order isomorphism of $\mathcal{S}$ onto its dual space. Then $\|\phi\| \cdot \|\phi^{-1}\| \geq 2$.

Proof. Let I denote the identity element of $\mathcal{S}$ and let $\delta_0 = \phi(I)$. Choose $H = H^* \in \mathcal{S}$ that is not in the span of I. Since δ_0 is positive, $\delta_0(H) \in \mathbb{R}$. Replacing H by $H - \delta_0(H)I$ we may assume that $\delta_0(H) = 0$. Now let $\delta_1 = \phi(H)$, which is a self-adjoint functional on $\mathcal{S}$. Set $M = \inf\{r : rI \geq H\}$ and set $m = \sup\{rI : H \geq rI\}$. Since H is not a multiple of I, it follows that $m < M$. For any real numbers a, b we will have that $\|aI + bH\| = max\{|a + bM|, |a + bm|\}$ and that $aI + bH \geq 0$ if and only if $min\{a + bM, a + bm\} \geq 0$. Since ϕ is a complete order isomorphism, $a\delta_0 + b\delta_1$ is completely positive if and only if $min\{a + bM, a + bm\} \geq 0$.

Now note that $\|MI - H\| = M - m = \|H - mI\|$ and that $MI - H \geq 0$, $H - mI \geq 0$, and so $M\delta_0 - \delta_1$ and $\delta_1 - m\delta_0$ are both completely positive. Let $\delta_1(I) = s$. The complete positivity of these last two maps, implies that $\|M\delta_0 - \delta_1\| = (M\delta_0 - \delta_1)(I) = M - s \geq 0$ and that $\|\delta_1 - m\delta_0\| = (\delta_1 - m\delta_0)(I) = s - m \geq 0$. Hence, $m \leq s \leq M$. Finally,

$$\|\phi\| \cdot \|\phi^{-1}\| \geq max\{\frac{\|MI - H\|}{\|M\delta_0 - \delta_1\|}, \frac{\|H - mI\|}{\|\delta_1 - m\delta_0\|}\} = max\{\frac{M - m}{M - s}, \frac{M - m}{s - m}\} \geq 2.$$

This last inequality follows by observing that the minimum of this maximum over s occurs when $s = (M + m)/2$. □

To construct SOH, we consider the finite-dimensional case, the extension to infinite-dimension case is standard. We use a few facts that are implicitly contained in Pisier's book [30, Exercise 7.2]. Fix a Hilbert space of dimension n and let $\{e_i\}$ be an orthonormal basis. Asume that $OH(n) \subseteq B(\mathcal{H})$ is a completely isometric inclusion, so that e_i are identified with operators. Let

$$H_i = \begin{bmatrix} 0 & e_i \\ e_i^* & 0 \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H}),$$

so that the H_i's are self-adjoint operators.

Given matrices, we have that

$$\|\sum_i A_i \otimes H_i\| = max\{\|\sum_i A_i \otimes e_i\|, \|\sum_i A_i \otimes e_i^*\|\} =$$
$$max\{\|\sum_i A_i \otimes \overline{A_i}\|^{1/2}, \|\sum_i A_i^* \otimes A_i^t\|^{1/2}\} = \|\sum_i A_i \otimes e_i\|.$$

This last equality follows since $A^t \otimes B^t = (A \otimes B)^t$ and so,

$$\|\sum_i A_i^* \otimes A_i^t\| = \|(\sum_i \overline{A_i} \otimes A_i)^t\| = \|\sum_i \overline{A_i} \otimes A_i\| = \|\sum_i A_i \otimes e_i\|^2.$$

Note in particular, we have that $\|\sum_i A_i \otimes e_i\| = \|\sum_i A_i^* \otimes e_i\| = \|\sum_i A_i^t \otimes e_i\|$.

Thus, the map $e_i \mapsto H_i$ is a complete isometry and we have that $OH(n)$ is also the span of these self-adjoint elements. The particular form of these self-adjoint operators will be useful in the sequel.

For notational convenience we let H_0 denote the identity operator on $\mathcal{H} \oplus \mathcal{H}$.

Definition 3.4. We let $SOH(n) \subseteq B(\mathcal{H} \oplus \mathcal{H})$ denote the $(n+1)$-dimensional operator system that is the span of the set $\{H_i : 0 \leq i \leq n\}$.

We now examine the norm and order structure on $SOH(n)$.

Proposition 3.5. Let $A_i \in M_m, 0 \leq i \leq n$. Then the following are equivalent:

- $\sum_{i=0}^n A_i \otimes H_i$ is positive,
- $A_0 \otimes H_0 - \sum_{i=1}^n A_i \otimes H_i$ is positive,
- $A_0 \in M_m^+$, $A_i = A_i^*, 1 \leq i \leq n$ and $-A_0 \otimes \overline{A_0} \leq \sum_{i=1}^n A_i \otimes \overline{A_i} \leq +A_0 \otimes \overline{A_0}$, in $M_m \otimes M_m = M_{m^2}$.

Proof. Let $U = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right] \in B(\mathcal{H} \oplus \mathcal{H})$, which is unitary. Note that $U^*H_0U = H_0$ and $U^*H_iU = -H_i, 1 \leq i \leq n$, from which the equivalence of the first two statements follows.

Adding the first two equations shows that $A_0 \geq 0$. Since a positive element must be self-adjoint it follows that $A_i = A_i^*, 1 \leq i \leq n$.

To see the final equations, first assume that A_0 is positive and invertible. Then $\sum_{i=0}^n A_i \otimes H_i$ is positive iff $(A_0 \otimes H_0)^{-1/2}(\sum_{i=0}^n A_i \otimes H_i)(A_0 \otimes H_0)^{-1/2}$ is positive which is iff $I_m \otimes H_0 + \sum_{i=1}^n B_i \otimes H_i$ is positive, where $B_i = A_0^{-1/2} A_i A_0^{-1/2}$. As operators on $\mathcal{H} \oplus \mathcal{H}$, we have that

$$\begin{bmatrix} I_{\mathcal{H}} & \sum_i B_i \otimes e_i \\ \sum_i B_i \otimes e_i & I_{\mathcal{H}} \end{bmatrix}$$

is positive.

This last equation is equivalent to requiring that the (1,2)-entry of this operator matrix is a contraction and hence, $\| \sum_i B_i \otimes \overline{B_i} \| \leq 1$. But since these matrices are self-adjoint, this is equivalent to

$$-I_m \otimes I_m \leq \sum_i B_i \otimes \overline{B_i} \leq I_m \otimes I_m.$$

Conjugating this last result by $A_0^{1/2} \otimes \overline{A_0^{1/2}}$ yields the desired inequality.

When A_0 is not invertible, one first considers $A_0 + rI_m, r > 0$ and then lets $r \to 0$. This completes the proof. $\square$

We now consider the matrix-ordered dual of $SOH(n)$. To this end we let $\delta_i \in SOH(n)^d, 0 \leq i \leq n$ denote the linear functionals such that $\delta_i(H_j) = \delta_{i,j}, 0 \leq i, j \leq n$.

Theorem 3.6. *The map $\kappa : SOH(n) \to SOH(n)^d$ defined by $\kappa(H_i) = \delta_i$, $0 \leq i \leq n$,* *is a complete order isomorphism that satisfies*

$$\|\sum_{i=0}^{n} A_i \otimes \delta_i\|_{cb} \le \|\sum_{i=0}^{n} A_i \otimes H_i\| \le 2\|\sum_{i=0}^{n} A_i \otimes \delta_i\|_{cb}$$

for any matrices $A_0, ..., A_n \in M_m$ *and any* m *and* $\|\kappa\|_{cb} \cdot \|\kappa^{-1}\|_{cb} = 2$.

Proof. First, we prove that κ is completely positive. Keeping the notation from the last proof, assume that $\sum_{i=0}^{n} A_i \otimes H_i$ is positive. We must prove that the map $\Phi : SOH(n) \to M_m$ given by $\Phi(X) = \sum_{i=0}^{n} A_i \otimes \delta_i(X)$ is completely positive. Assume that A_0 is invertible and define B_i as above. Let $P = \sum_{i=0}^{n} P_i \otimes H_i \in M_q(SOH(n))^+$. We must show that

$$\Phi^{(q)}(P) = \sum_{i=0}^{n} A_i \otimes P_i \in (M_n \otimes M_q)^+.$$

Assuming that P_0 is also invertible, we set $Q_i = P_0^{-1/2} P_i P_0^{-1/2}$. By the last proposition, we have that $\|\sum_{i=1}^{n} B_i \otimes e_i\| \le 1$ and $\|\sum_{i=1}^{n} Q_i \otimes e_i\| \le 1$. Hence, by the self-duality of $OH(n)$, we have that $\|\sum_{i=1}^{n} B_i \otimes Q_i\|_{M_m \otimes M_q} \le 1$. Using the fact that all these matrices are self-adjoint, yields

$$-I_m \otimes I_q \le \sum_{i=1}^{n} B_i \otimes Q_i \le +I_m \otimes I_q.$$

Thus, $I_m \otimes I_q + \sum_{i=1}^{n} B_i \otimes Q_i \ge 0$, which after conjugation by $A_0^{1/2} \otimes P_0^{1/2}$ yields that $\Phi^{(q)}(P) \ge 0$.

Conversely, if $\Phi = \sum_{i=0}^{n} A_i \otimes \delta_i \in M_m(SOH(n)^d)$ is completely positive, then it follows that $A_0 \ge 0$, and that $A_i = A_i^*, 1 \le i \le n$. Taking B_i's as before, we have that $\Psi = I_m \otimes \delta_0 + \sum_{i=1}^{n} B_i \otimes \delta_i$ is a unital completely positive map and hence completely contractive. Applying this map to any element $\sum_i C_i \otimes e_i \in M_q(OH(n))$ of norm less than one, yields that $\| \sum_{i=1}^n B_i \otimes C_i \| \leq 1$. Thus, by self-duality of $OH(n)$ we have that $\| \sum_{i=1}^n B_i \otimes \overline{B_i} \| \leq 1$. Hence, $-I_m \otimes I_m \leq \sum_{i=1}^n B_i \otimes \overline{B_i} \leq +I_m \otimes I_m$ and the Proposition 3.5 implies that $\sum_{i=0}^n A_i \otimes H_I$ is positive. Thus, κ is a complete order isomorphism.

We now consider the norm inequalities. Let $X = \sum_{i=0}^n A_i \otimes H_i$, set $\Phi = \sum_{i=0}^n A_i \otimes \delta_i$ and assume that $\|X\|_{SOH(n)} \leq 1$. Here, the matrices A_i are no longer necessarily self-adjoint. We then have that

$$0 \leq \begin{bmatrix} I_{\mathcal{H}} \otimes I_m & X \\ X^* & I_{\mathcal{H}} \otimes I_m \end{bmatrix} = \begin{bmatrix} I_m & A_0 \\ A_0^* & I_m \end{bmatrix} \otimes H_0 + \sum_{i=1}^n \begin{bmatrix} 0 & A_i \\ A_i^* & 0 \end{bmatrix} \otimes H_i.$$

From the fact that κ is completely positive, it follows that

$$\begin{bmatrix} I_m & A_0 \\ A_0^* & I_m \end{bmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{bmatrix} 0 & A_i \\ A_i^* & 0 \end{bmatrix} \otimes \delta_i = \begin{bmatrix} I_m \otimes \delta_o & \sum_{i=0}^n A_i \otimes \delta_i \\ \sum_{i=0}^n A_i^* \otimes \delta_i & I_m \otimes \delta_0 \end{bmatrix} = \begin{bmatrix} \Psi & \Phi \\ \Phi^* & \Psi \end{bmatrix},$$

and Ψ is a unital completely positive map. Hence, $\|\Phi\|_{cb} \leq 1$ and it follows that $\|\kappa^{(m)}(X)\|_{cb} \leq \|X\|$ for any $X \in M_m(SOH(n))$ and any m.

Conversely, assume that $\Phi = \sum_{i=0} A_i \otimes \delta_i$. To prove the other inequality, it will be enough to assume that $\|\Phi\|_{cb} \leq 1$ and show that $\|X\|_{SOH(n)} \leq 2$.

Since $\|\Phi||_{cb} \leq 1$, there exist unital completely positive maps $\Psi_j \colon SOH(n) \to M_m, j = 1, 2$ such that the map $\Gamma = \left[\begin{smallmatrix} \Psi_1 & \Phi \\ \Phi^* & \Psi_2 \end{smallmatrix}\right] : SOH(n) \to M_{2m}$ is completely positive. Writing $\Psi_j = \sum_{i=0}^n C_i^j \otimes \delta_i$, we have that $\Gamma = \sum_{i=0} \left[\begin{smallmatrix} C_i^1 & A_i \\ A_i^* & C_i^2 \end{smallmatrix}\right] \otimes \delta_i$. Moreover, since the maps Ψ_j are unital, $C_0^1 = C_0^2 = I_m$. By the Proposition and the fact that κ is a complete order isomorphism, we assert that complete positivity of Γ implies that

$\Gamma_1 = \begin{bmatrix} I_m & A_0 \\ A_0^* & I_m \end{bmatrix} \otimes \delta_0 - \sum_{i=1}^n \begin{bmatrix} C_i^1 & A_i \\ A_i^* & C_i^2 \end{bmatrix} \otimes \delta_i$ is completely positive. Adding $\Gamma + \Gamma_1$, and using the positivity, yields that $\|A_0\| \leq 1$.

Next, if we let Γ_2 be the completely positive map obtained by conjugating the coefficient matrices of Γ_1 by the unitary $U = \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix}$, we find that $\Gamma_2 = \begin{bmatrix} I_m & -A_0 \\ -A_0^* & I_m \end{bmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{bmatrix} -C_i^1 & A_i \\ A_i^* & -C_i^2 \end{bmatrix} \otimes \delta_i$. The average $1/2(\Gamma+\Gamma_2) = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} \otimes \delta_0 + \sum_{i=1}^n \begin{bmatrix} 0 & A_i \\ A_i^* & 0 \end{bmatrix} \otimes \delta_i$ is a unital completely positive map.

Using that κ is a complete order isomorphism and replacing the δ_i's by H_i's, yields that $\| \sum_{i=1}^n A_i \otimes H_i\| \leq 1$. Hence,

$$\| \sum_{i=0}^n A_i \otimes H_i \| \leq \|A_0 \otimes H_0\| + \| \sum_{i=1}^n A_i \otimes H_i\| \leq 2$$

and the desired inequality follows.

Finally, we have that $\|\kappa\|_{cb} \leq 1$ and $\|\kappa^{-1}\|_{cb} \leq 2$, so that $\|\kappa\| \cdot \|\kappa^{-1}\|_{cb} \leq 2$ and so we must have equality by Proposition 3.3. □

By the above results we see that, among all self-dual operator systems, the operator systems $SOH(n)$ acheive the minimal cb-condition number of 2. However, this does not uniquely characterize these spaces. In fact, M_2 is another self-dual operator system that attains this minimum.

One other example is ℓ_2^∞, but it is not hard to see that this operator system is unitally, completely order-isomorphic to $SOH(1)$. Indeed, consider the map $\phi\colon SOH(1) \to l_2^\infty$, $\phi(H_0) = (1,1)$ and $\phi(H_1) = (1,-1)$. By Proposition 3.5, $\alpha H_0 + \beta H_1 \geq 0$ if and only if $\alpha H_0 - \beta H_1 \geq 0$, if and only if $-\alpha^2 \leq \beta^2 \leq \alpha^2$. An easy calculation shows that these conditions are equivalent to $\alpha \geq 0$ and $-\alpha \leq \beta \leq \alpha$, if and only if $\phi(\alpha H_0 + \beta H_1) = (\alpha+\beta, \alpha-\beta)$ is positive. Since l_2^∞ is a C*-algebra, ϕ^{-1}, hence ϕ, is a unital complete order isomorphism.

It would be interesting to try to characterize the self-dual operator systems that attain this minimal cb-condition number.

3.3 Some Structure Results for SOH

In [30], $OH(n)$ is defined in a basis-free fashion. In this section we show that $SOH(n)$ is also independent of basis, which leads to proving that every quotient and operator subsystem of $SOH(n)$ is unitally completely order-isomorphic to some $SOH(m)$. We also derive a few properties of $SOH(n)$ that will be useful in the sequel. To avoid ambiguity, whenever we work with $SOH(n)$ and $SOH(m)$, we denote $H_i^{(n)}$ and $H_j^{(m)}$, respectively, the basis elements H_i as defined in the last section.

Proposition 3.7. Let $1 \leq n \leq m$ and let $\{\vec{u}_i = (u_{ij}) \in \mathbb{R}^m\}_{i=1}^n$ be an orthonormal set. Then the map $\Phi\colon SOH(n) \to SOH(m)$ defined by $\Phi(I) = I$ and $\Phi(H_i^{(n)}) := \sum_{j=1}^m u_{ij} H_j^{(m)}$ is a unital complete order inclusion.

Proof. Consider self-adjoint $\sum_{i=0}^n A_i \otimes H_i^{(n)} \in M_p \otimes SOH(n)$. Let $B_0 = A_0$ and for $j = 1, \ldots, n$, let $B_j = \sum_{i=1}^n u_{ij} A_i$. Then $\sum_{i=0}^n A_i \otimes \Phi(H_i)$ is

$$B_0 \otimes H_0^{(m)} + \sum_{i=1}^n A_i \otimes (\sum_{j=1}^m u_{ij} H_j^{(m)}) = B_0 \otimes I + \sum_{j=1}^m B_j \otimes H_j^{(m)}.$$

Observe that $B_j = B_j^*$; and by orthonormality of the $\vec{u}_i$'s,

$$\sum_{j=1}^m B_j \otimes \overline{B_j} = \sum_{j=1}^m \left(\sum_{i,k=1}^n u_{ij} u_{kj} \right) A_i \otimes \overline{A_k} = \sum_{i,k=1}^n \left(\sum_{j=1}^m u_{ij} u_{kj} \right) A_i \otimes \overline{A_k}$$
$$= \sum_{i,k=1}^n \delta_{i,k} A_i \otimes \overline{A_k} = \sum_{i=1}^n A_i \otimes \overline{A_i}.$$

Therefore, $\{A_i\}_{i=0}^n$ satisfies the third condition in Proposition 3.5 if and only if $\{B_j\}_{j=0}^m$ satisfies the same condition, proving that $\sum_{i=0}^n A_i \otimes H_i^{(n)} \geq 0$ if and only if $\sum_{i=0}^n A_i \otimes \Phi(H_i^{(n)}) \geq 0$; this is equivalent to Φ being a unital complete order inclusion. □

Corollary 3.8. Let $U = [u_{ij}] \in M_n(\mathbb{R})$ be an orthonormal matrix and set $K_0 = H_0$, $K_i = \sum_{j=1}^n u_{ij} H_j$. Then the map $\Phi\colon SOH(n) \to SOH(n)$ given by $\Phi(H_0) = K_0$ and $\Phi(H_i) = K_i$ is a unital complete order isomorphism.

Given $n \leq m$, it is now clear that $SOH(n) \subset_{ucoi} SOH(m)$. We will see that every operator subsystem of $SOH(m)$ is necessarily $SOH(n)$.

Corollary 3.9. If $\mathcal{T}$ is an operator subsystem of $SOH(m)$ of dimension $n+1$, then $\mathcal{T}$ is unitally completely order-isomorphic to $SOH(n)$.

Proof. Let $\{K_0 = I, K_i = K_i^*\colon i = 1, \ldots, n\}$ be a basis for $\mathcal{T}$. Without loss of generality, we assume for each $i = 1, \ldots, n$, $K_i = \sum_{j=1}^m a_{ij} H_j^{(m)}$ for some $a_{ij} \in \mathbb{R}$. We first claim that the vectors $\vec{a_i} = (a_{ij}) \in \mathbb{R}^m$ are linearly independent. For if not, then $\vec{a_i} = \sum_{k=1,k\neq i}^n \lambda_k \vec{a}_k$, for some i, leading to $K_i = \sum_{j=1}^m \sum_{k=1,k\neq i}^n \lambda_k H_j^{(m)}$, which contradicts our assumption.

Now consider the n-dimensional subspace of $\mathbb{R}^m$ spanned these $\vec{a_i}$'s. Pick an orthonormal basis $\{\vec{u}_i = (u_{ij}) \in \mathbb{R}^m\}_{i=1}^n$ for this subspace and define $\Phi\colon SOH(n) \to SOH(m)$ by $\Phi(I) = I$ and $\Phi(H_i^{(n)}) = \sum_{j=1}^m u_{ij} H_j^{(m)}$. By the last proposition, Φ is a complete order inclusion. It remains to check that the image of Φ is $\mathcal{T}$. Since every $\vec{a_i} = \sum_{k=1}^n \lambda_k^i \vec{u}_k$, for each K_i we can write

$$K_i = \sum_{j=1}^m a_{ij} H_j^{(m)} = \sum_{j=1}^m \sum_{k=1}^n \lambda_k^i u_{kj} H_j^{(m)} = \sum_{k=1}^n \lambda_k^i \Phi(H_j^{(n)}),$$

proving that $\Phi(SOH(n)) = \mathcal{T}$. Consequently $\mathcal{T} \cong_{ucoi} SOH(n)$ via Φ. □

Hence every operator subsystem of $SOH(n)$ is again of the same form. The next result characterizes quotients of $SOH(n)$ based on self-duality.

Proposition 3.10. Let $\mathcal{J}$ be a non-trivial self-adjoint subspace of $SOH(n)$. Then the following are equivalent:

1. $\mathcal{J}$ is the kernel of some unital, completely positive map with domain $SOH(n)$.
2. There exist $m < n$ and a surjective unital completely positive map $\phi\colon SOH(n) \to SOH(m)$ such that $\mathcal{J} = \ker(\phi)$.
3. There is unital completely positive map ϕ on $SOH(n)$ for which $\mathcal{J} = \ker(\phi)$.

Proof. The direction (2) $\implies$ (3) $\implies$ (1) is obvious. Now assume (1) and let $q\colon SOH(n) \to SOH(n)/\mathcal{J}$ be the canonical quotient map. Then $q^d\colon (SOH(n)/\mathcal{J})^d \to SOH(n)^d = SOH(n)$ is a unital complete order embedding [10]. Since $\mathcal{J}$ is non-trivial, $(SOH(n)/\mathcal{J})^d$ has dimension $m < n$ and by the last corollary $(SOH(n)/\mathcal{J})^d \cong SOH(m)$. By duality, $SOH(n)/\mathcal{J} \cong SOH(m)^d = SOH(m)$. $\square$

In Section 8 of [15], it is shown that the coproduct of two operator systems $\mathcal{S}$ and $\mathcal{T}$ can be obtained by operator system quotients. Namely, $\mathcal{S} \oplus_1 \mathcal{T} \cong_{ucoi} (\mathcal{S} \oplus \mathcal{T})/\mathcal{J}$, where $\mathcal{J} = \mathbb{C}(1_{\mathcal{S}}, -1_{\mathcal{T}})$.

Proposition 3.11. For any $p \in \mathbb{N}$, let $H_0^{(p)}, ..., H_p^{(p)}$ denote the canonical basis for $SOH(p)$. Then for any $n, m \in \mathbb{N}$, the map $\phi : SOH(n) \oplus SOH(m) \to SOH(n+m)$ defined by $\phi(H_j^{(n)}) = H_j^{(n+m)}, 0 \leq j \leq n$ and $\phi(H_j^{(m)}) = \begin{cases} H_0^{(n+m)}, & j = 0 \\ H_{n+j}^{(n+m)}, & j > 0 \end{cases}$ induces a unital completely positive map $\Phi : SOH(n) \oplus_1 SOH(m) \to SOH(n+m)$, but this map is not an order isomorphism.

Proof. It is easy to check that the restriction of ϕ to each direct summand is a unital completely positive map. Hence, Φ is a unital completely positive map by the universal property of the coproduct.

To see that Φ is not an order isomorphism, it suffices to show that $SOH(1) \oplus_1 SOH(1) \neq SOH(2)$. Suppose the contrary and consider the positive element $P = \sqrt{2}H_0^{(2)} + H_1^{(2)} + H_2^{(2)}$ in $SOH(2)$. Then there must be positive numbers a and b such that $(aH_0^{(1)} + H_1^{(1)})$ and $(bH_0^{(1)} + H_1^{(1)})$ are positive in $SOH(1)$ and sum to P in $SOH(2)$. By Proposition 3.5, each of a^2 and b^2 is greater than 1; however $a+b = \sqrt{2}$ implies that $2ab \leq 0$, contradicting a and b are positive. $\square$

Question 3.12. The last result brings up a natural question. Is there a notion of Hilbert coproduct, or 2-coproduct, of operator systems in general? If so, does it naturally identify $SOH(n) \oplus_2 SOH(m) \cong SOH(n+m)$?

Proposition 3.13. Let $\mathcal{S}$ be an operator system and $\{h_i \colon h_i = h_i^*, ||h_i|| \leq 1\}_{i=1}^n \subset \mathcal{S}$. Then there is $r > 0$ such that the map $\phi\colon SOH(n) \to \mathcal{S}$ given by $H_0 \mapsto r1_{\mathcal{S}}$, $H_i \mapsto h_i$ is completely positive.

Proof. Choose $r > n^{1/2}$ and suppose $A_0 \otimes H_0 + \sum_{i=1}^n A_i \otimes H_i$ is positive in $M_m \otimes SOH(n)$. We will show that $rA_0 \otimes 1_{\mathcal{S}} + \sum_{i=1}^n A_i \otimes h_i$ is positive. First assume $A_0 > 0$ is invertible. We claim

$$\begin{bmatrix} rA_0 \otimes 1_{\mathcal{S}} & \sum_{i=1}^n A_i \otimes h_i \\ \sum_{i=1}^n A_i^* \otimes h_i^* & rA_0 \otimes 1_{\mathcal{S}} \end{bmatrix}$$

is positive in $M_{2m} \otimes SOH(n)$, which is equivalent to

$$r^{-1}||\sum_{i=1}^n A_0^{-1/2} A_i A_0^{-1/2} \otimes h_i||_{M_m \otimes \mathcal{S}} \leq 1.$$

Write $B_i = A_0^{-1/2} A_i A_0^{-1/2}$, then by Proposition 3.5, $||\sum_{i=1}^n B_i \otimes \overline{B_i}|| \le 1$. Now embed $\mathcal{S} \subset B(\mathcal{H})$ and regard $h_i \otimes \overline{h_i}$ as an operator in $B(\mathcal{H} \otimes \overline{H})$. Then by a version of the Cauchy-Schwarz inequality due to Haagerup [13, Lemma 2.4],

$$\begin{aligned} r^{-1} || \sum_{i=1}^n B_i \otimes h_i ||_{M_m \otimes \mathcal{S}} &\le r^{-1} || \sum_{i=1}^n B_i \otimes \overline{B_i} ||_{M_{2m}}^{1/2} \cdot || \sum_{i=1}^n h_i \otimes \overline{h_i} ||_{B(\mathcal{H} \otimes \overline{\mathcal{H}})}^{1/2} \\ &\le r^{-1} n^{1/2} < 1. \end{aligned}$$

Hence, the above matrix is positive as claimed. By $*$-conjugating it by $[1,1]$, we deduce that $2(rA_0 \otimes 1_{\mathcal{S}} + \sum_{i=1}^n A_i \otimes h_i)$ is positive. When A_0 is not invertible, apply the standard $A_0 + \varepsilon I_m$ argument as in the proof of Proposition 3.5. Consequently, ϕ is completely positive. $\square$

Corollary 3.14. In the previous settings, if $\mathcal{S}$ is an operator system, then the map $\theta \colon \mathcal{S}^d \to SOH(n)$ by $\theta(f) = rf(1_{\mathcal{S}})H_0 + \sum_{i=1}^n f(h_i)H_i$ is completely positive.

Proof. The dual map $\phi^d \colon \mathcal{S}^d \to SOH(n)^d$, $\phi^d(f)(H_i) = f \circ \phi(H_i)$, is completely positive. Let $\kappa \colon SOH(n) \to SOH(n)^d$ be the map $h_i \mapsto \delta_i$ as in Theorem 3.6. Then by self-duality of $SOH(n)$, the map $\kappa^{-1} \circ \phi^d \colon \mathcal{S}^d \to SOH(n)$ is completely positive and an easy calculation shows that $\kappa^{-1} \circ \phi^d(f) = \theta(f)$. $\square$

Chapter 4

The γ_{soh} Tensor Product

Some of the important Banach space tensor products arise via factorization of bounded maps through Hilbert space. For operator spaces, Pisier in [29, 30] also constructed the γ_{oh} tensor product of operator spaces via factorization of complete contractive maps through the operator Hilbert space OH. In Chapter 2, we already had a similar result on the maximal tensor product through the matrix algebras M_n. In this chapter, we are seeking an analogue to γ_{oh} for operator systems, via factorization of completely positive maps through the operator Hilbert system SOH.

We first construct this tensor product with finite-dimensional operator systems. We denote it by γ_{soh} and examine some of its properties. More importantly, we show that γ_{soh} is distinct from the other tensor products found in [17, 18], and establish some nuclearity-related results about γ_{soh}. Then we provide a general method to extend functorial tensor product of finite-dimensional operator systems to the infinite-dimensional case. Consequently, we can extend γ_{soh} and generalize Theorem 2.12 to infinite-dimensional operator systems.

4.1 The γ_{soh} Tensor Product

In Chapter 2, it is shown that the positive cone of the maximal tensor product of finite-dimensional operator systems, $\mathcal{S} \otimes_{\max} \mathcal{T}$, can be identified with the completely positive maps from $\mathcal{S}^d$ to $\mathcal{T}$ that factor through M_n approximately; equivalently these are the nuclear maps. Motivated by this characterization, we will construct the γ_{soh} tensor product similarly by using $M_p(SOH(n))$ instead of M_n. We show that $\phi_1 \otimes \phi_2 \colon \mathcal{S}_1 \otimes_{\gamma_{soh}} \mathcal{S}_2 \to \mathcal{T}_1 \otimes_{\gamma_{soh}} \mathcal{T}_2$ is completely positive whenever $\phi_i \colon \mathcal{S}_i \to \mathcal{T}_i$ is completely positive. We prove that γ_{soh} is a functorial and symmetric tensor product structure in the category of finite-dimensional operator systems. We also prove that γ_{soh} is a distinct tensor product from many of the functorial tensors studied in [8, 17, 18].

Definition 4.1. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We say that $\hat{u} : \mathcal{S}^d \to \mathcal{T}$ **factors through SOH approximately**, provided there exist nets of completely positive maps $\phi_\lambda \colon \mathcal{S}^d \to M_{p_\lambda}(SOH(n_\lambda))$ and $\psi_\lambda \colon M_{p_\lambda}(SOH(n_\lambda)) \to \mathcal{T}$ such that $\psi_\lambda \circ \phi_\lambda$ converges to $\hat{u}$ in the point-norm topology.

Definition 4.2 (The γ_{soh}-cone)**.** Let $\mathcal{S}$ and $\mathcal{T}$ be finite-dimensional operator systems. Define

$$\mathcal{C}_1^\gamma(\mathcal{S}, \mathcal{T}) = \{u \in \mathcal{S} \otimes \mathcal{T} \colon \hat{u} \text{ factors through } SOH \text{ approximately}\}.$$

For $u = [u_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T})$, we regard $\hat{u} = [\hat{u_{ij}}]$ as a map from $\mathcal{S}^d$ to $M_n(\mathcal{T})$. Thus there is no confusion to define $\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$ in $M_n(\mathcal{S} \otimes \mathcal{T})$. We denote the triple $(\mathcal{S} \otimes \mathcal{T}, \{\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ by $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.

Proposition 4.3. The collection $\{\mathcal{C}_n^\gamma(\mathcal{S}, \mathcal{T})\}$ is a compatible family of proper cones of $\mathcal{S} \otimes \mathcal{T}$.

Proof. Since $\mathcal{C}_n^\gamma(\mathcal{S},\mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$, it suffices to check that $\mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$ is a proper cone. It is obvious that $\mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$ is closed under positive scalar multiplication. Let $u_1, u_2 \in \mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$, so there are nets of completely positive maps $\phi_{\lambda_k}, \psi_{\lambda_k}$, where $k = 1, 2$ such that $\lim_\lambda \psi_{\lambda_k} \circ \phi_{\lambda_k} = \hat{u_k}$ in the point-norm topology.

Consider the directed set $\Lambda = \{(\lambda_1, \lambda_2)\}$ with the natural ordering. For each $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, regard $M_{p_\lambda} = M_{p_{\lambda_1}} \oplus M_{p_{\lambda_2}}$ as the 2-by-2 block and let $n_\lambda = \max\{n_{\lambda_1}, n_{\lambda_2}\}$. Note that every completely positive map on $SOH(n_{\lambda_k})$, $k = 1, 2$, can be extended naturally on $SOH(n_\lambda)$. Thus without loss of generality we may assume that ϕ_{λ_k} maps into $M_{p_\lambda} \otimes SOH(n_\lambda)$ and ψ_{λ_k} has domain $M_{p_\lambda} \otimes SOH(n_{\lambda_k})$.

Thus, for each $\lambda = (\lambda_1, \lambda_2)$, we take $M_{p_\lambda}(SOH(n_\lambda))$, with completely positive maps $\phi_\lambda = \phi_{\lambda_1} \oplus \phi_{\lambda_2}$ and $\psi_\lambda = \psi_{\lambda_1} \oplus \psi_{\lambda_2}$. It remains to check that $\psi_\lambda \circ \phi_\lambda$ converges to $(\widehat{u_1 + u_2})$ in the point-norm topology. Indeed, given $f \in \mathcal{S}^d$ and $\varepsilon > 0$, by assumption there exist μ_1 and μ_2 so that $||\hat{u_k}(f) - \psi_{\lambda_k} \circ \phi_{\lambda_k}(f)|| < \frac{\varepsilon}{2}$, for $\lambda_k > \mu_k$. Thus if $\mu = (\mu_1, \mu_2)$ and $\lambda > \mu$, then

$$||(\widehat{u_1 + u_2})(f) - (\psi_\lambda \circ \phi_\lambda)(f)|| \leq \sum_{k=1}^{2} ||\hat{u_k}(f) - (\psi_k \circ \phi_k)(f)|| < \varepsilon$$

shows that $u_1 + u_2$ is in $\mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$.

Next we verify compatability. Let $u = [u_{ij}] \in \mathcal{C}_n^\gamma(\mathcal{S},\mathcal{T})$ with $\hat{u}$ factors through SOH approximately via nets ψ_λ and ϕ_λ. Write $A = [a_{kl}] \in M_{m,n}$, and write $w = AuA^* \in M_m(\mathcal{S} \otimes \mathcal{T})$. We claim that $\hat{w}$ also factors through SOH approximately via the nets $(\theta_A \circ \psi_\lambda)$ and ϕ_λ, where $\theta_A \colon M_n(\mathcal{T}) \to M_m(\mathcal{T})$ by $B \mapsto ABA^*$ is completely positive. To this end, note that by writing $w = [\sum_{k,l}^n a_{i,k}u_{k,l}\overline{a_{l,j}}]_{i,j=1}^m$, for each $f \in \mathcal{S}^d$

$$\begin{aligned}\hat{w}(f) &= \left[\sum_{k,l=1}^{n} (a_{i,k}\widehat{u_{k,l}\overline{a_{l,j}}})(f)\right]_{i,j=1}^{m} \\ &= \left[\sum_{k,l=1}^{n} a_{i,k}\hat{u}_{k,l}(f)\overline{a_{l,j}}\right]_{i,j=1}^{m} = A\hat{u}(f)A^* = (\theta_A \circ \hat{u})(f).\end{aligned}$$

Thus, for each $f \in \mathcal{S}^d$,

$$||\hat{w}(f) - \theta_A \circ \psi_\lambda \circ \phi_\lambda(f)|| = ||\theta_A \circ (\hat{u} - \psi_\lambda \circ \phi_\lambda)(f)|| \to 0.$$

Therefore, $\{\mathcal{C}_n^\gamma(\mathcal{S},\mathcal{T})\}$ is a compatible family of proper cones. □

Proposition 4.4. The unit $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an Archimedean matrix order unit for $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.

Proof. Again by identifying $\mathcal{C}_n^\gamma(\mathcal{S},\mathcal{T}) = \mathcal{C}_1^\gamma(\mathcal{S}, M_n(\mathcal{T}))$, it suffices to prove that $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an Archimedean order unit for $\mathcal{S} \otimes_\gamma \mathcal{T}$ on the ground level. Let $u \in \mathcal{S} \otimes \mathcal{T}$ be self-adjoint, we must find an $r > 0$ so that $r1_\mathcal{S} \otimes 1_\mathcal{T} - u$ is in $\mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$. Without loss of generality, we may assume $u = \sum_{i=1}^n x_i \otimes y_i$, where $x_i = x_i^*$ and $y_i = y_i^*$. By Proposition 3.13 and Corollary 3.14, there exist $r_1, r_2 > 0$ such that the map $\phi\colon \mathcal{S}^d \to SOH(n)$ by $\phi(f) = r_1 f(1_\mathcal{S})H_0 - \sum_{i=1}^n f(x_i)H_i$, and $\psi\colon SOH(n) \to \mathcal{T}$ by $\psi(H_0) = r_2 1_\mathcal{T}$, $\psi(H_i) = y_i$ are completely positive. Choose $r = r_1 r_2$, then

$$\psi(\phi(f)) = r_1 r_2 f(1_\mathcal{S})1_\mathcal{T} - \sum_{i=1}^n f(x_i)y_i = (r1_\mathcal{S}\widehat{\otimes 1_\mathcal{T}} - u)(f)$$

shows that $(r1_\mathcal{S}\widehat{\otimes 1_\mathcal{T}} - u)$ factors through $SOH(n)$ exactly. Consequently, $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an order unit for $\mathcal{S} \otimes_\gamma \mathcal{T}$.

Finally suppose $u = \sum_{i=0}^n x_i \otimes y_i \in \mathcal{S} \otimes \mathcal{T}$ and for each $\varepsilon > 0$, $u_\varepsilon = u + \varepsilon(1_\mathcal{S} \otimes 1_\mathcal{T}) \in$

$\mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$. For each ε, there is a net of completely positive maps $\phi_{\lambda_\varepsilon}$ and $\psi_{\lambda_\varepsilon}$ such that

$$\begin{tikzcd} \mathcal{S}^d \arrow[rr, "\hat{u_\varepsilon}"] \arrow[dr, "\phi_{\lambda_\varepsilon}"'] & & \mathcal{T} \\ & M_{p_{\lambda_\varepsilon}}(SOH(n_{\lambda_\varepsilon})) \arrow[ur, "\psi_{\lambda_\varepsilon}"'] & \end{tikzcd}$$

and $||\hat{u_\varepsilon}(f) - (\psi_{\lambda_\varepsilon} \circ \phi_{\lambda_\varepsilon})(f)|| \to 0$, for each $f \in \mathcal{S}^d$.

Hence for each fixed ε, by finite dimensionality of $\mathcal{S}^d$, there exist a sufficiently large $k > \frac{1}{\varepsilon}$ and a pair of completely positive maps $\phi_{\lambda_{(\varepsilon,k)}}$ and $\psi_{\lambda_{(\varepsilon,k)}}$ from the net $(\psi_{\lambda_\varepsilon} \circ \phi_{\lambda_\varepsilon})$, such that $||\hat{u_\varepsilon}(f) - (\psi_{\lambda_{(\varepsilon,k)}} \circ \phi_{\lambda_{(\varepsilon,k)}})(f)|| < \frac{1}{k}$, for every $||f|| \leq 1$.

Consider the directed set Λ consisting of (ε, k) subject to the above condition, and order it by $(\varepsilon,k) \leq (\varepsilon',k')$ if and only if $\varepsilon' \leq \varepsilon$ and $k' \geq k$. Now we claim that $(\psi_\lambda \circ \phi_\lambda)_{\lambda\in\Lambda}$ converges to $\hat{u}$ in the point-norm topology. Given $f \in \mathcal{S}^d$ with $||f|| \leq 1$, for each $m > 0$, consider for $\varepsilon > \frac{1}{2m}$ and those $\lambda = (\varepsilon, k)$,

$$\begin{aligned} ||\hat{u}(f) - (\psi_\lambda \circ \phi_\lambda)(f)|| &= ||\hat{u}(f) - \hat{u_\varepsilon}(f) + \hat{u_\varepsilon}(f) - (\psi_\lambda \circ \phi_\lambda)(f)|| \\ &\leq ||\hat{u}(f) - \hat{u_\varepsilon}(f)|| + ||\hat{u_\varepsilon}(f) - (\psi_\lambda \circ \phi_\lambda)(f)|| \\ &< \frac{1}{2m} + \frac{1}{2m}. \end{aligned}$$

Therefore, $\hat{u}$ factors through $M_p(SOH(n))$ approximately and $u \in \mathcal{C}_1^\gamma(\mathcal{S},\mathcal{T})$. Consequently, $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an Archimedean matrix order unit. $\square$

Definition 4.5. The triple $(\mathcal{S}\otimes\mathcal{T}, \mathcal{C}_n^\gamma(\mathcal{S},\mathcal{T}), 1_\mathcal{S} \otimes 1_\mathcal{T})$ is an operator system, and we denote it by $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$.

Theorem 4.6. *The γ_{soh} tensor product defines a functorial operator system tensor product structure in the category of finite-dimensional operator systems.*

Proof. Let $P \in M_n(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$. Note that by regarding $\mathcal{S} = (\mathcal{S}^d)^d$ and $P\colon \mathcal{S}^d \to M_n$, then $(P \hat{\otimes} Q)\colon \mathcal{S}^d \to M_{nm}(\mathcal{T})$ maps f to $P(f) \otimes Q$. Moreover, $P \hat{\otimes} Q$ factors through $M_n \otimes SOH(1)$ via

$$\begin{tikzcd} \mathcal{S}^d \arrow[rr, "(P\hat{\otimes}Q)"] \arrow[dr, "P\otimes H_0"'] & & M_{nm}(\mathcal{T}) \\ & M_n \otimes SOH(1) \arrow[ur, "I_n\otimes Q"'] & \end{tikzcd}$$

Therefore, $P \otimes Q \in \mathcal{C}^{\gamma}_{nm}(\mathcal{S}, \mathcal{T})$.

For the functorial property, let $\rho\colon \mathcal{S} \to \mathcal{V}$ and $\kappa\colon \mathcal{T} \to \mathcal{W}$ be completely positive maps between finite-dimensional operator systems, and let $u \in \mathcal{S} \otimes_\gamma \mathcal{T}$ be positive. Thus $\hat{u}$ factors through $M_p(SOH(n))$ approximately via some ϕ_λ and ψ_λ. Let $w = (\rho \otimes \kappa)(u) \in \mathcal{V} \otimes \mathcal{W}$. Notice this diagram

$$\begin{tikzcd} \mathcal{V}^d \arrow[rr, "\hat{w}"] \arrow[d, "\rho^d"'] & & \mathcal{W} \\ \mathcal{S}^d \arrow[rr, "\hat{u}"] \arrow[dr, "\phi_\lambda"'] & & \mathcal{T} \arrow[u, "\kappa"'] \\ & M_{p_\lambda}(SOH(n_\lambda)) \arrow[ur, "\psi_\lambda"'] & \end{tikzcd}$$

commutes and the maps are all completely positive. Indeed, if $w = \sum_{i=1}^n \rho(x_i) \otimes \kappa(y_i)$, where $u = \sum_{i=1}^n x_i \otimes y_i$, then for each $f \in \mathcal{V}^d$,

$$\begin{aligned} \hat{w}(f) &= \sum_{i=0}^n f(\rho(x_i))\kappa(y_i) = (\kappa \circ \hat{u} \circ \rho^d)(f) \\ &= \lim_\lambda (\kappa \circ \psi_\lambda) \circ (\phi_\lambda \circ \rho^d)(f). \end{aligned}$$

Therefore, $\hat{w}$ also factors through $M_p(SOH(n))$ approximately and $w \in (\mathcal{V} \otimes_{\gamma_{soh}} \mathcal{W})^+$.

For $u = [u_{ij}] \in M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$, in the same vein we regard $\hat{u}\colon \mathcal{S} \to M_n(\mathcal{T})$. Then by replacing κ by $\kappa \otimes I_n$ and $\mathcal{W}$ by $M_n(\mathcal{W})$ we deduce that $\hat{u}$ factors through SOH approximately. Consequently $\rho \otimes \kappa$ is completely positive and the γ_{soh} tensor product is functorial. $\square$

In [8], the ess tensor product $\mathcal{S} \otimes_{ess} \mathcal{T}$ arises by the inclusion in $C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$, where $C_e^*(\mathcal{S})$ is the enveloping C^*-algebra of $\mathcal{S}$. It was yet to know whether this tensor product is functorial. Recently in [12, Proposition 3.2], Gupta and Luthra proved that the ess tensor product is not functorial. This allows us to distinguish γ_{soh} from ess.

Corollary 4.7. The γ_{soh} tensor product is not the ess tensor product.

We deduce further properties of the γ_{soh} tensor product.

Proposition 4.8. The γ_{soh} tensor is symmetric.

Proof. If $u \in (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$, then by self-duality of $SOH(n)$ we see that

$$\begin{tikzcd} \mathcal{T}^d \arrow[rr, "\hat{u}^d"] \arrow[dr, "\psi_\lambda^d"'] & & \mathcal{S}^{dd} = \mathcal{S} \\ & M_{p_\lambda}(SOH(n_\lambda)) \arrow[ur, "\phi_\lambda^d"'] & \end{tikzcd}$$

commutes. Indeed, if $u = \sum_{i=1}^n x_i \otimes y_i$, then for $g \in \mathcal{T}^d$ and $f \in \mathcal{S}^d$,

$$(\hat{u}^d)(g)(f) = g(\hat{u}(f)) = \sum_{i=1}^n g(y_i) f(x_i) = \hat{u}(f)(g).$$

Hence, $\hat{u}$ factors through $M_p(SOH(n))$ approximately if and only if $\hat{u}^d$ does. At the matrix level, we identify $M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+ = (\mathcal{S} \otimes_{\gamma_{soh}} M_n(\mathcal{T}))^+ = (M_n(\mathcal{T}) \otimes_{\gamma_{soh}} \mathcal{S})^+ = (\mathcal{T} \otimes_{\gamma_{soh}} M_n(\mathcal{S}))^+ = M_n(\mathcal{T} \otimes_{\gamma_{soh}} \mathcal{S})^+$. This shows that $x \otimes y \mapsto y \otimes x$ is a complete order isomorphism from $\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$ onto $\mathcal{T} \otimes_{\gamma_{soh}} \mathcal{S}$. $\square$

In [17], there are some tensor products constructed using the injective envelope. These come from the identifications, $\mathcal{S} \otimes_{el} \mathcal{T} \subset_{coi} \mathcal{I}(\mathcal{S}) \otimes_{\min} \mathcal{T}$, where $\mathcal{I}(\mathcal{S})$ is the injective envelope of $\mathcal{S}$, and likewise for $\mathcal{S} \otimes_{er} \mathcal{T}$. It turns out that the *el* and *er*-tensor products are not symmetric.

Corollary 4.9. The γ_{soh} tensor product is neither the *er* nor the *el* tensor product.

Theorem 4.10. *The γ_{soh} tensor product is not the maximal tensor product. In particular, for $n \geq 2$, $SOH(n) \otimes_{\gamma_{soh}} SOH(n) \neq SOH(n) \otimes_{\max} SOH(n)$.*

Proof. By self-duality of $SOH(n)$, it suffices to show that $SOH(n)^d \otimes_{\gamma_{soh}} SOH(n) \neq SOH(n)^d \otimes_{\max} SOH(n)$. Consider the element $u = \sum_{i=0}^{n} \delta_i \otimes H_i$. Note that $\hat{u}$ is in fact the identity map on $SOH(n)$ and factors through SOH trivially, so $u \in (SOH(n)^d \otimes_{\gamma_{soh}} SOH(n))^+$.

On the other hand, if $u \in SOH(n)^d \otimes_{\max} SOH(n)$ were positive, then by Theorem 2.12, $\hat{u}$ factors through the matrix algebras approximately. Thus $SOH(n)$ has CPFP and by Theorem 1.18, $SOH(n)$ must be $(\min, \max)$-nuclear. It follows that $SOH(n)$ is $(c, \max)$-nuclear and by Proposition 1.20, it is unitally completely order-isomorphic to a finite-dimensional C*-algebra. However it follows that $OH(n)$ could be completely isometrically represented on a finite-dimensional Hilbert space and is hence 1-exact, contradicting Pisier's result [30]. Therefore, u is not positive in $SOH(n)^d \otimes_{\max} SOH(n)$. Consequently the two operator systems are not completely isomorphic. □

We have seen that γ_{soh} is likely a new tensor product. The next natural question is to ask which operator systems are nuclear with respect to γ_{soh}. The following result characterizes $(\min, \gamma_{soh})$-nuclearity by identifying the matricial cone structures of the minimal tensor product to completely positive maps. Following this characterization, we are able to deduce that γ_{soh} is also distinct from the commuting tensor product, and that γ_{soh} is a not self-dual tensor product.

Theorem 4.11. *Let $\mathcal{S}$ and $\mathcal{T}$ be finite-dimensional operator systems. Then $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T}$ if and only if every completely positive map from $\mathcal{S}^d$ to $\mathcal{T}$ factors through SOH approximately.*

Proof. By Theorem 1.14, $\mathcal{S} \otimes_{\min} \mathcal{T} =_{ucoi} (\mathcal{S}^d \otimes_{\max} \mathcal{T}^d)^d$, whose cone $(\mathcal{S}^d \otimes_{\max} \mathcal{T}^d)^{d,+}$ is in one-to-one correspondence to $CP(\mathcal{S}^d, \mathcal{T})$. Hence $\phi \in CP(\mathcal{S}^d, \mathcal{T})$ if and only if $\phi = \hat{u}$ for some $u \in (\mathcal{S} \otimes_{\min} \mathcal{T})^+$; and $\hat{u}$ factors through SOH approximately if and only if $u \in (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$. Consequently, $(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = (\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$ if and only if every completely positive $\phi\colon \mathcal{S}^d \to \mathcal{T}$ admits such a factorization. At the matrix level, we identify $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ with $(\mathcal{S} \otimes_\tau M_n(\mathcal{T}))^+$ for $\tau = \min, \gamma_{soh}$; then the result follows from the base case. $\square$

Corollary 4.12. $SOH(n)$ is $(\min, \gamma_{soh})$-nuclear.

Corollary 4.13. The γ_{soh} tensor product is not the commuting tensor product.

Proof. If $\gamma_{soh} = c$, then $SOH(n) \otimes_{\min} SOH(n) = SOH(n) \otimes_{\gamma_{soh}} SOH(n) = SOH(n) \otimes_c SOH(n)$. By self-duality of $SOH(n)$, $SOH(n) \otimes_{\min} SOH(n)^d = SOH(n) \otimes_c SOH(n)^d$. By Proposition 1.20, it follows that $SOH(n)$ is C*-nuclear, thus exact. This is a contradiction to Pisier's result that $OH(n)$ is not exact [30], as in the proof of the last theorem. $\square$

Corollary 4.14. The γ_{soh} tensor product is not self-dual.

Proof. Suppose γ_{soh} is self-dual; that is, $(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^d = \mathcal{S}^d \otimes_{\gamma_{soh}} \mathcal{T}^d$. Then $SOH(n) \otimes_{\min} SOH(n)^d = SOH(n) \otimes_{\gamma_{soh}} SOH(n)^d$ and by dualizing one obtains $SOH(n)^d \otimes_{\max}$

$SOH(n) = SOH(n)^d \otimes_{\gamma_{soh}} SOH(n)$. Again by Proposition 1.20, $SOH(n)$ is C*-nuclear, which is a contradiction. □

4.2 Extension to the Infinite-Dimensional Case

In this section we show that every functorial tensor product structure defined on the category of finite-dimensional operator systems can be extended to infinite-dimensional operator systems. We also prove that this extension preserves symmetry, injectivity, and projectivity. Therefore, the γ_{soh} tensor product defined in the previous section, as well as the characterization of maximal tensor product given by Theorem 2.12, can now be extended to infinite-dimensional operator systems.

Given an operator system $\mathcal{S}$, we denote the collection of finite-dimensional operator subsystems of $\mathcal{S}$ by $\mathcal{F}(\mathcal{S})$.

Definition 4.15. Let τ be a functorial tensor product structure on the category of finite-dimensional operator systems. We define $\tilde{\tau}$ on the category of operator systems in the following way: Given $\mathcal{S}$ and $\mathcal{T}$, for each $n \in \mathbb{N}$, define the family of proper cones

$$\mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T}) := \bigcup_{E \in \mathcal{F}(\mathcal{S}), F \in \mathcal{F}(\mathcal{T})} M_n(E \otimes_\tau F)^+.$$

Theorem 4.16. *$\tilde{\tau}$ defines a functorial tensor product structure on the category of operator systems.*

Proof. Let us denote $\mathcal{C}_n^{\tilde{\tau}} = \mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T})$. We first claim that it defines a matrix-ordering on $\mathcal{S} \otimes \mathcal{T}$. It is trivial that $\mathcal{C}_n^{\tilde{\tau}}$ is a proper cone for each n. To show that this is a matrix-ordering, we first check that for each $m, n \in \mathbb{N}$, $A \in M_{n,m}(\mathbb{C})$, $A^*\mathcal{C}_n^{\tilde{\tau}}A \subset C_m^{\tilde{\tau}}$.

Since every $B \in \mathcal{C}_n^{\tilde{\tau}}$ belongs to $M_n(E \otimes_\tau F)^+$, for some $E \in \mathcal{F}(\mathcal{S})$ and $F \in \mathcal{F}(\mathcal{T})$, we have $A^*BA \in \mathcal{C}_m(E \otimes_\alpha F) \subset \mathcal{C}_m^{\tilde{\tau}}$.

To see that $1 \otimes 1$ is an Archimedean matrix order unit for $(\mathcal{S} \otimes \mathcal{T}, \mathcal{C}_n^{\tilde{\tau}})$, consider $A \in M_n(\mathcal{S} \otimes \mathcal{T})$ such that for each $\varepsilon > 0$, $A_\varepsilon = \varepsilon(1 \otimes 1) \otimes I_n + A \in \mathcal{C}_n^{\tilde{\tau}}$. By definition, there exist $E_\varepsilon \in \mathcal{F}(\mathcal{S})$ and $F_\varepsilon \in \mathcal{F}(\mathcal{T})$ for which $A_\varepsilon \in M_n(E_\varepsilon \otimes_\tau F_\varepsilon)^+$. Let $E = \bigcap_{\varepsilon>0} E_\varepsilon \in \mathcal{F}(\mathcal{S})$ and $F = \cap_{\varepsilon>0} F_\varepsilon \in \mathcal{F}(\mathcal{T})$, then by functorial property of τ, for each $\varepsilon > 0$, $M_n(E \otimes_\tau F)^+ \subsetneq M_n(E_\varepsilon \otimes_\tau F_\varepsilon)^+$. Finally, since $E \otimes_\tau F$ defines an operator system, as $\varepsilon \to 0$ we see that $A_\varepsilon \to A \in M_n(E \otimes_\tau F)^+$. Consequently, $1 \otimes 1$ is an Archimedean matrix order unit; and $(\mathcal{S} \otimes \mathcal{T}, \mathcal{C}_n^{\tilde{\tau}}, 1 \otimes 1)$ is an operator system.

It remains to show the (T2) and (T3) axioms. Given $P = (p_{ij}) \in M_n(\mathcal{S})^+$ and $Q = (q_{st}) \in M_m(\mathcal{T})^+$, by choosing E and F to be the spans of p_{ij}'s and q_{st}'s, we have $P \otimes Q \in M_{nm}(E \otimes_\tau F)^+$. This shows that $M_n(\mathcal{S})^+ \otimes M_m(\mathcal{T})^+ \subset \mathcal{C}_{nm}^{\tilde{\tau}}$. For (T3), we show further that it is functorial. Suppose $\phi\colon \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi\colon \mathcal{T}_1 \to \mathcal{T}_2$ are completely positive maps. If $A \in \mathcal{C}_k^{\tilde{\tau}}$, then there are $E_1 \in \mathcal{F}(\mathcal{S})$ and $F_1 \in \mathcal{F}(\mathcal{T})$ such that $A \in M_k(E_1 \otimes_\tau F_1)^+$. Let E_2 and F_2 denote the ranges of ϕ and ψ, respectively. By functorial property of τ, the map $\phi \otimes \psi|_{E_1 \otimes_\tau F_1} : E_1 \otimes_\tau F_1 \to E_2 \otimes_\tau F_2$ is completely positive. In particular, $(\phi \otimes \psi)^{(k)}(A) \in M_k(E_2 \otimes_\tau F_2)^+$. Therefore, $\phi \otimes \psi$ is completely positive and $\tilde{\tau}$ is functorial. □

Proposition 4.17. $\tilde{\tau}$ preserves injectivity, symmetry, and projectivity.

Proof. Let τ be injective, $\mathcal{S}_1 \subset \mathcal{S}$ and $\mathcal{T}_1 \subset \mathcal{T}$ be operator subsystems, and $A \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+ \cap M_n(\mathcal{S}_1 \otimes \mathcal{T}_1)$. By definition, $A \in M_n(E \otimes_\tau F)^+$ for some finite-dimensional operator subsystems $E \subset \mathcal{S}$ and $F \subset \mathcal{T}$. Hence $E \cap \mathcal{S}_1$ and $F \cap \mathcal{T}_1$ are finite-dimensional operator subsystems of $\mathcal{S}_1$ and $\mathcal{T}_1$ respectively. By injectivity of τ,

$$A \in M_n(E \otimes_\tau F)^+ \cap M_n(\mathcal{S}_1 \otimes \mathcal{T}_1) = M_n((E \cap \mathcal{S}_1) \otimes_\tau (F \cap \mathcal{T}_1))^+.$$

This shows that $A \in M_n(\mathcal{S}_1 \otimes_{\tilde{\tau}} \mathcal{T}_1)^+$, and $\mathcal{S}_1 \otimes_{\tilde{\tau}} \mathcal{T}_1$ is complete order included in $\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T}$, proving $\tilde{\tau}$ is injective.

Let τ be symmetric, and $\phi\colon \mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T} \to \mathcal{T} \otimes_{\tilde{\tau}} \mathcal{S}$ be the map $x \otimes y$ to $y \otimes x$. If $u \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+$, then $u \in M_n(E \otimes_\tau F)^+$, for some finite-dimensional E and F; so $\phi^{(n)}(u) \in M_n(F \otimes_\tau E)^+ \subset M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})^+$ and $\tilde{\tau}$ is symmetric.

Suppose τ is projective, and $q\colon \mathcal{S} \to \mathcal{V}$ and $\rho\colon \mathcal{T} \to \mathcal{W}$ are complete quotient maps. We claim that every $U \in M_n(\mathcal{V} \otimes_{\tilde{\tau}} \mathcal{W})^+$ can lift to a positive $\tilde{U} \in M_n(\mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T})$. Since $U \in M_n(X \otimes_\tau Y)^+$, for some $X \in \mathcal{F}(\mathcal{V})$ and $Y \in \mathcal{W}(\mathcal{T})$, using projectivity of τ, there is $\tilde{U} \in M_n(E \otimes_\tau F)^+$ for which $E \in \mathcal{F}(\mathcal{S})$, $F \in \mathcal{F}(\mathcal{T})$ and $q \otimes \rho(\tilde{U}) = U$. Therefore, $\tilde{\tau}$ is projective. $\square$

Remark 4.18. *We remark that $\tilde{\tau}$ indeed extends τ. If $\mathcal{S}$ and $\mathcal{T}$ are finite-dimensional, then $\mathcal{C}_n^{\tilde{\tau}}(\mathcal{S}, \mathcal{T}) = M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ by functorial property of τ, thus $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\tilde{\tau}} \mathcal{T}$.*

Lemma 4.19. *Let τ be a symmetric tensor product structure. Then τ is left projective (resp. injective) if and only if it is right projective (resp. injective), if and only if it is projective (resp. injective).*

Proof. Let $q\colon \mathcal{S} \to \mathcal{R}$ be a complete quotient map. Then this commuting diagram

$$\begin{tikzcd}
\mathcal{S} \otimes_\tau \mathcal{T} \arrow[r] \arrow[d, "q\otimes id"'] & \mathcal{T} \otimes_\tau \mathcal{S} \arrow[d, "id\otimes q"] \\
\mathcal{R} \otimes_\tau \mathcal{T} & \mathcal{T} \otimes_\tau \mathcal{R} \arrow[l]
\end{tikzcd}$$

asserts the equivalent condition. Similarly, if $\mathcal{R}$ is a operator subsystem of $\mathcal{S}$, then

$$\begin{array}{ccc} \mathcal{S} \otimes_\tau \mathcal{T} & \longrightarrow & \mathcal{T} \otimes_\tau \mathcal{S} \\ {\scriptstyle \iota\otimes id}\downarrow & & \downarrow{\scriptstyle id\otimes\iota} \\ \mathcal{R} \otimes_\tau \mathcal{T} & \longleftarrow & \mathcal{T} \otimes_\tau \mathcal{R} \end{array}$$

shows that τ is left injective if and only if it is right injective. □

Consequently, given functorial τ on finite-dimensional operator systems, there is no ambiguity to say that τ defines a tensor product structure on arbitrary operator systems. Now the construction of γ_{soh} and characterization of max in Theorem 2.12 can be extended in the infinite-dimensional case. In particular, the cone $M_n(\mathcal{S} \otimes_{\gamma_{soh}} \mathcal{T})^+$ is precisely the set of $u \in \mathcal{S} \otimes M_n(\mathcal{T})$ so that $\hat{u}\colon E^d \to M_n(F)$ factors through SOH approximately, for some $E \in \mathcal{F}(\mathcal{S})$ and $F \in \mathcal{F}(\mathcal{T})$.

Some questions about γ_{soh} remain. We do not know if it is injective or projective. By the lemma above, it suffices to check these properties on one side. We do not yet know if γ_{soh} is distinct from any of the symmetric tensors that arise from two-sided inclusions into the maximal tensor products of the injective envelope or the C*-envelope.

Chapter 5

The Paulsen System

Given a concrete operator space $V \subset B(\mathcal{H})$, Paulsen [26] proved that there is operator system $\mathcal{S}_V$, now known as the **Paulsen system** of V, given by

$$\mathcal{S}_V = \left\{ \begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix} : \lambda, \mu \in \mathbb{C}, X, Y \in V \right\} \subset M_2(B(\mathcal{H})).$$

This operator system has the property that for any complete contraction $\varphi\colon V \to \mathcal{A}$ into a C*-algebra $\mathcal{A}$, the induced map $\Phi\colon \mathcal{S}_V \to M_2(\mathcal{A})$ given by

$$\Phi\left(\begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix} \right) = \begin{bmatrix} \lambda I & \varphi(X) \\ \varphi(Y)^* & \mu I \end{bmatrix}$$

is a unital completely positive map. Many results on complete contractions and completely positive maps can then be extended easily to completely bounded maps by the use of the Paulsen system. Nevertheless, little is known about its matrix-ordered dual $\mathcal{S}_V^d$.

We begin the study by a characterization of the states of $\mathcal{S}_V$. By this characterization, we deduce that $\mathcal{S}_V^d$ with an appropriate order unit is an operator system. This motivates us to study the relation between $\mathcal{S}_V^d$ and $\mathcal{S}_{V^*}$, where V^* denotes the operator space dual of V. Moreover, we examine a natural operator system quotient of the $\mathcal{S}_V$ by a certain kernel $\mathcal{J}$, and deduce that $(\mathcal{S}_V/\mathcal{J})^d$ can be regarded as an operator subsystem of $\mathcal{S}_V^d$. We also prove that $\mathcal{S}_V/\mathcal{J}$, when equipped with the two operator space quotient norms obtained from either the operator space or the operator system structures of $\mathcal{S}_V$, are completely bounded-isomorphic.

5.1 States on the Paulsen System

It is more convenient to study the matricial order structure of $\mathcal{S}_V$ by a faithful representation. Henceforth, let $\mathcal{A}$ be a unital C*-algebra represented on a Hilbert space $\mathcal{H}$ and V be an operator space in $\mathcal{A}$. We begin by recalling a simple, yet useful, relation between $M_2(\mathcal{A})^+$ and inner product in Chpater 3 of [26]. This tool becomes handy when we study the relation between $M_n(\mathcal{S}_V)^+$ and $M_n(V)$.

Lemma 5.1. *Suppose $\mathcal{A}$ is a unital C*-algebra faithfully represented on $\mathcal{H}$. A 2-by-2 matrix $\left[\begin{smallmatrix} A & T \\ T^* & B \end{smallmatrix}\right] \in M_2(B(\mathcal{H}))^+$ if and only if for every $x, y \in \mathcal{H}$, $|\langle Tx, y\rangle|^2 \leq \langle Ax, x\rangle\langle By, y\rangle$. In particular, for any $X \in \mathcal{A}$, $||X|| \leq 1$ if and only if $\left[\begin{smallmatrix} I & X \\ X^* & I \end{smallmatrix}\right] \in M_2(\mathcal{A})^+$.*

Corollary 5.2. The matrix $Q = \left[\begin{smallmatrix} \lambda I & X \\ Y^* & \mu I \end{smallmatrix}\right]$ is positive in $\mathcal{S}_V$ if and only if $\lambda, \mu \geq 0$, $Y = X$, and $||X|| \leq \sqrt{\lambda\mu}$.

Proposition 5.3. Given $Q = \left[\left[\begin{smallmatrix} \lambda_{ij} I & X_{ij} \\ Y_{ij}^* & \mu_{ij} I \end{smallmatrix}\right]_{ij}\right]$ in $M_n(\mathcal{S}_V)$, write $\Lambda = [\lambda_{ij}], U = [\mu_{ij}] \in M_n$, and $X = [X_{ij}] \in M_n(V)$. Then Q is unitarily equivalent to $\left[\begin{smallmatrix} \Lambda & X \\ [Y_{ji}]^* & U \end{smallmatrix}\right]$ in

$M_2(M_n(\mathcal{A}))$. Moreover, if both Λ and U are strictly positive, then Q is positive if and only if $||\Lambda^{-1/2}XU^{-1/2}|| \leq 1$.

Proof. The first assertion is known as the canonical shuffle; it can be seen using tensor notation:

$$\begin{aligned} Q &= \sum_{i,j}^{n}(\lambda_{ij}I \otimes E_{11} + X_{ij} \otimes E_{12} + Y_{ij}^* \otimes E_{21} + \mu_{ij}I \otimes E_{22}) \otimes E_{i,j} \\ &= (\sum_{i,j}^{n} \lambda_{ij}I \otimes E_{ij}) \otimes E_{11} + (\sum_{i,j}^{n} X_{ij} \otimes E_{ij}) \otimes E_{12} \\ &\quad + (\sum_{i,j}^{n} Y_{ij}^* \otimes E_{ij}) \otimes E_{21} + (\sum_{i,j}^{n} \mu_{ij}I \otimes E_{ij}) \otimes E_{22}, \end{aligned}$$

which is unitarily equivalent to the above matrix. Hence it is positive if and only $\begin{bmatrix} \Lambda & X \\ [Y_{ij}^*] & U \end{bmatrix}$ is positive in $M_2(M_n(\mathcal{A}))$; if and only if $\Lambda, U \geq 0$ and $X_{ij} = Y_{ji}^*$. If Λ and U are strictly positive, multiplying $\left[\begin{smallmatrix} \Lambda & X \\ X^* & U \end{smallmatrix}\right]$ on both the left and the right by $\Lambda^{-1/2} \oplus U^{-1/2}$ yields that $\begin{bmatrix} \Lambda^{-1/2}XU^{-1/2} & I \\ I & \Lambda^{-1/2}XU^{-1/2} \end{bmatrix}$. By the previous lemma, it is positive if and only if $||\Lambda^{-1/2}XU^{-1/2}|| \leq 1$. □

Note that if either Λ or U is 0, then Q must be 0 by the first lemma. To avoid triviality, in the sequel when we say $\left[\begin{smallmatrix} \Lambda & X \\ X^* & U \end{smallmatrix}\right]$ is positive, we always assume $\Lambda, U > 0$.

Corollary 5.4. Let $Q = \left[\begin{bmatrix} \lambda_{ij}I & X_{ij} \\ Y_{ij}^* & \mu_{ij}I \end{bmatrix}_{ij}\right]$ in $M_n(\mathcal{S}_V)^+$. Let $f \colon V \to M_m$ be completely bounded, then the matrix

$$\begin{bmatrix} ||f||_{cb}\Lambda \otimes I_m & f^{(n)}(X) \\ f^{(n)}(X)^* & ||f||_{cb}U \otimes I_m \end{bmatrix}$$

is positive in $M_2(M_{nm})$.

Proof. Observe that if $A, B \in M_n(\mathbb{C})$, then

$$f^{(n)}(AXB) = (A \otimes I_m) f^{(n)}(X)(B \otimes I_m).$$

As in the proof of last lemma, $||\Lambda^{-1/2} X U^{-1/2}|| \leq 1$, so

$$\begin{aligned} ||(\Lambda^{-1/2} \otimes I_m) f^{(n)}(X)(U^{-1/2} \otimes I_m)||_{M_{nm}} \\ &= ||f^{(n)}(\Lambda^{-1/2} X U^{-1/2})|| \\ &\leq ||f^{(n)}|| \cdot ||\Lambda^{-1/2} X U^{-1/2}|| \\ &\leq ||f^{(n)}|| = ||f||_{cb}. \end{aligned}$$

Hence, the matrix

$$\begin{bmatrix} ||f||_{cb}(I_n \otimes I_m) & (\Lambda^{-1/2} \otimes I_n) f^{(n)}(X)(U^{-1/2} \otimes I_n) \\ [(\Lambda^{-1/2} \otimes I_n) f^{(n)}(X)(U^{-1/2} \otimes I_n)]^* & ||f||_{cb}(I_n \otimes I_m) \end{bmatrix}$$

is in $M_2(M_{nm})^+$. Now $*$-conjugate this matrix by $(\Lambda^{1/2} \otimes I_m) \oplus (U^{1/2} \otimes I_m)$, then we obtain the desired result. □

The above results establish a clear picture of the matricial order structure of $\mathcal{S}_V$ in general. It is thus natural to ask about the states of $\mathcal{S}_V$ and their relation to linear functionals on V.

Theorem 5.5. *Let s be a state on $\mathcal{S}_V$. Then there exist $t \in [0,1]$ and a linear functional f on V, with $||f|| \leq \sqrt{t(1-t)}$ such that*

1. $s\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = t$ *and* $s\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right) = 1 - t.$

2. $s\left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\right) = f(X)$ *and* $s\left(\begin{bmatrix} 0 & 0 \\ X^* & 0 \end{bmatrix}\right) = \overline{f(X)}$.

Conversely, for every such pair t and f, the map $\Phi_{t,f}$ given by the above formulae defines a state on $\mathcal{S}_V$.

Proof. We first prove the converse. Since s is a state on $\mathcal{S}_V$, the fact that $I_{\mathcal{A}} \otimes E_{11}$ and $I_{\mathcal{A}} \otimes E_{22}$ are positive and sum up to $I_{M_2(\mathcal{A})}$ asserts that there is a $t \in [0,1]$ satisfying condition (1). Define $f\colon V \to \mathbb{C}$ by $f(X) = s(X \otimes E_{12})$. We claim that $||f|| \leq \sqrt{t(1-t)}$.

By Corollary 5.2, for each $||X|| \leq \sqrt{\lambda\mu}$,

$$s\left(\begin{bmatrix} \lambda I & X \\ X^* & \mu I \end{bmatrix}\right) = \lambda t + \mu(1-t) + f(X) + \overline{f(X)} \geq 0.$$

By choosing a unimodular $e^{i\theta}$ so that $f(e^{i\theta}X) = -|f(X)|$, we deduce that $2|f(X)| \leq \lambda t + \mu(1-t)$, for each $||X|| \leq \sqrt{\lambda\mu}$. Let Y be the unit vector of X, we see that

$$|f(Y)| \leq \frac{1}{2\sqrt{\lambda\mu}}(\lambda t + \mu(1-t)),$$

for every unit vector Y, and every $\lambda, \mu > 0$. Let h_t be the function of $(\lambda, \mu) \in [0,\infty) \times [0,\infty)$ in the above inequality, hence $||f|| \leq h_t(\lambda,\mu)$. By elementary Calculus, for $0 < t < 1$, h_t attains its global minimum value $\sqrt{t(1-t)}$ at $\lambda t = \mu(1-t)$. If $t = 0, 1$, then $\inf h_t = 0$. Therefore, $||f|| \leq \sqrt{t-(1-t)}$ as claimed.

Now suppose $t \in (0,1)$, $||f|| \leq \sqrt{t(1-t)}$ and define

$$\Phi_{t,f}\left(\begin{bmatrix} \lambda I & X_1 \\ X_2^* & \mu I \end{bmatrix}\right) = t\lambda + (1-t)\mu + f(X_1) + \overline{f(X_2)}.$$

By Lemma 5.1, $\begin{bmatrix} \lambda I & X \\ X^* & \mu I \end{bmatrix} \geq 0$ if and only if $||X|| \leq \sqrt{\lambda\mu}$ and $\lambda, \mu \geq 0$. Thus,

$$\begin{aligned} t\lambda + (1-t)\mu + f(X) + \overline{f(X)} &\geq t\lambda + (1-t)\mu - 2|f(X)| \\ &\geq t\lambda + (1-t)\mu - 2\sqrt{\lambda\mu}\sqrt{t(1-t)} \\ &= (\sqrt{t\lambda} + \sqrt{(1-t)\mu})^2 \geq 0. \end{aligned}$$

Consequently, $\Phi_{t,f}$ defines a state on $\mathcal{S}_V$. □

Corollary 5.6. For any $t \in (0,1)$, $f \in V^*$ with $||f|| < \sqrt{t(1-t)}$, the state $\Phi_{t,f}$ is faithful.

Proof. Let $\begin{bmatrix} \lambda I & X \\ X^* & \mu I \end{bmatrix} > 0$, and consider

$$\Phi_{t,f}\left(\begin{bmatrix} \lambda I & X \\ X^* & \mu I \end{bmatrix}\right) = t\lambda + (1-t)\mu + 2\mathrm{Re}(f(X)).$$

By the same argument in the last proof, this quantity is strictly greater than $t\lambda + (1-t) - 2\sqrt{\lambda u}\sqrt{t(1-t)} = (\sqrt{t\lambda} - \sqrt{(1-t)\mu})^2 \geq 0$, whenever $\lambda, \mu > 0$; so $\Phi_{t,f}$ is faithful. □

From the proof above, we see that by maximizing the function $G(t) = \sqrt{t(1-t)}$

at $t = \frac{1}{2}$, indeed any f for which $||f|| = r \leq \frac{1}{2}$ yields a state $\Phi_{1/2,f}$. Moreover, there correspond $t_1 < \frac{1}{2} < t_2$ such that $G(t_i) = r$. Hence, every such f produces a subset $S(f) = \{\Phi_{t,f} \colon t \in [t_1, t_2]\}$ of the state space on $\mathcal{S}_V$. The state space of $\mathcal{S}_V$ is thus the union of all such $S(f)$, where $||f|| \leq \frac{1}{2}$.

5.2 $\mathcal{S}_{V^*}$ and $\mathcal{S}_V^d$

For an operator system $\mathcal{S}$, recall that $\mathcal{S}^d$ is a matrix-ordered space by the identification $M_n(\mathcal{S}^d) \cong L(\mathcal{S}, M_n)$. At the matrix level, the involution is given by $[f_{ij}]^*(X) = [f_{ij}(X^*)]^* = [\overline{f_{ji}(X^*)}]$. We then equip $M_n(\mathcal{S}^d)$ with following matricial order structure:

$$M_n(\mathcal{S}^d)^+ = \{F = [f_{ij}] \mid F \colon \mathcal{S} \to M_n,$$
$$F(x) = [f_{ij}(x)] \text{ is completely postive}\}.$$

By Theorem 1.2, if $\mathcal{S}$ is finite-dimensional, there exists a state s on $\mathcal{S}$ that serves as an Archimedean matrix order unit for $\mathcal{S}^d$, hence making $\mathcal{S}^d$ into an operator system. However, the result fails in infinite-dimensional case because of the absence of such state. In the case of the Paulsen system $\mathcal{S}_V$, regardless of dimension of V, it turns out that there always exists such a state on S_V.

Proposition 5.7. The mapping $tr \colon \mathcal{S}_V \to \mathbb{C}$ by

$$tr\left(\begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix}\right) = \lambda + \mu$$

is an Archimedean order unit for $\mathcal{S}_V^d$.

Proof. We first show that tr is a matrix order unit, let $h \in M_n(\mathcal{S}_V^d)$ be self-adjoint. By viewing $h\colon \mathcal{S}_V \to M_n$, then

$$h\left(\begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix}\right) = \lambda A + \mu B + f(X) + f(Y)^*,$$

where $A = h(I \otimes E_{11})$ and $B = h(I \otimes E_{22})$ are self-adjoint, and $f(X) = h(X \otimes E_{12})$ is bounded from V to M_n since h is bounded. Choose $r \geq 0$ such that $rI_n - A$ and $rI_n - B \geq ||f||_{cb} I_n$.

We first claim that $r(tr \otimes I_n) - h$ is a positive map from $\mathcal{S}_V$ to M_n. For each $\left[\begin{smallmatrix} \lambda I & X \\ X^* & \mu I \end{smallmatrix}\right] \in \mathcal{S}_V^+$, since $f(X) + f(X)^*$ is self-adjoint, by Corollary 5.2, $f(X) + f(X)^* \leq 2||f||\sqrt{\lambda\mu} I_n \leq 2||f||_{cb}\sqrt{\lambda\mu} I_n$. The following shows that

$$\begin{aligned}(r(tr \otimes I_n) - h)\left(\begin{bmatrix} \lambda I & X \\ X^* & \mu I \end{bmatrix}\right) &= \lambda(rI_n - A) + \mu(rI_n - B) - (f(X) + f(X)^*) \\ &\geq \lambda(rI_n - A) + \mu(rI_n - B) - 2||f||\sqrt{\lambda\mu} I_n \\ &\geq (\lambda + \mu - 2\sqrt{\lambda\mu})(||f||_{cb} I_n) \\ &= (\sqrt{\lambda} - \sqrt{\mu})^2(||f||_{cb} I_n) \geq 0.\end{aligned}$$

Hence, $r(tr \otimes I_n) - h$ is a positive map from $\mathcal{S}_V$ to M_n.

Now we will show that $r(tr \otimes I_n) - h$ is completely positive. Recall that a linear map $\phi\colon \mathcal{S} \to M_n$ is completely positive if and only if it is n-positive. Suppose $Q = \left[\left[\begin{smallmatrix} \lambda_{ij} I & X_{ij} \\ Y_{ij}^* & \mu I \end{smallmatrix}\right]_{ij}\right]$ is positive in $M_n(\mathcal{S}_V)$. Then

$$[(r(tr \otimes I_n) - h)(Q)]_{i,j=1}^n = [\lambda_{ij}(rI_n - A) + \mu_{ij}(rI_n - B) - (f(X_{ij}) + f(Y_{ij})^*)]_{i,j=1}^n.$$

By Lemma 5.3, the scalar matrices $\Lambda = [\lambda_{ij}], U = [\mu_{ij}] \geq 0$ and $Y_{ij} = X_{ji}$. Then the above matrix can be viewed as

$$\Lambda \otimes (rI_n - A) + U \otimes (rI_n - B) - [f(X_{ij}) + f(X_{ji})^*]_{i,j=1}^n. \tag{$\dagger$}$$

Note that $[f(X_{ij})]_{i,j=1}^n = f^{(n)}(X)$, so by our choice of r, we will be done if we show

$$\Lambda \otimes (||f||_{cb} I_n) + U \otimes (||f||_{cb} I_n) - f^{(n)}(X) - f^{(n)}(X)^*$$

is positive. Indeed, Corollary 5.4 asserts that the matrix

$$\begin{bmatrix} ||f||_{cb}\Lambda \otimes I_n & f^{(n)}(X) \\ f^{(n)}(X)^* & ||f||_{cb} U \otimes I_n \end{bmatrix}$$

is positive in $M_2(M_{n^2})$. By $*$-conjugating the above matrix by $\begin{bmatrix} I_n & I_n \end{bmatrix}$, we deduce that ($\dagger$) is positive. Consequently, $r(tr \otimes I_n) - h$ is completely positive, hence in $M_n(\mathcal{S}_V^d)^+$, and tr is a matrix order unit.

Finally it suffices to check that tr is Archimedean. Suppose $P = [P_{ij}] \in M_n(\mathcal{S}_V^d)$ such that for each $\varepsilon > 0$, the map $P_\varepsilon = P + \varepsilon(tr \otimes I_n)$ is completely positive from $\mathcal{S}_V$ to M_n. Let $[Q_{kl}] \in M_m(\mathcal{S}_V)^+$, where $Q_{kl} = \begin{bmatrix} \lambda_{kl} I & X_{kl} \\ Y_{kl}^* & \mu_{kl} I \end{bmatrix}$. Then,

$$[P_\varepsilon([Q_{kl}])] = [P([Q_{kl}])] + \varepsilon[(\lambda_{kl} + \mu_{kl})I_n]_{k,l=1}^m \in (M_m \otimes M_n)^+.$$

By letting $\varepsilon \to 0$ we see that $P^{(m)}([Q_{kl}]) \in (M_m \otimes M_n)^+$, so P is completely positive and belongs to $M_n(\mathcal{S}_V^d)^+$. Consequently, tr is an Archimedean matrix order unit. $\square$

Since any positive multiple of an Archimedean order unit is still Archimedean, $\frac{1}{2}tr$ is also Archimedean. Moreover, it is a faithful state on $\mathcal{S}_V$. Henceforth, we view $\mathcal{S}_V^d = (\mathcal{S}_V^d, \{M_n(\mathcal{S}_V^d)^+\}, \frac{1}{2}tr)$ as an operator system. We write V^* for the dual operator space of bounded linear functionals on V and embed V^* into $\mathcal{A}'$ for some unital C*-algebra $\mathcal{A}'$. We now turn to study the relation between $\mathcal{S}_{V^*}$ and $\mathcal{S}_V^d$. Recall that an element $T \in \mathcal{S}_{V^*}$ is of the form

$$T = \begin{bmatrix} aI & f \\ g^* & bI \end{bmatrix},$$

where $a, b \in \mathbb{C}$ and $f, g \in V^*$, where g^* denotes the operator adjoint of g in $\mathcal{A}'$.

Given $\phi \in \mathcal{S}_V^d$, as in the proof of Theorem 5.5, we decompose ϕ into four components and $a = \phi(1 \otimes E_{11})$, $b = \phi(1 \otimes E_{22})$, $f(X) = \phi(X \otimes E_{12})$, and $g(X) = \phi(X^* \otimes E_{21})$. Define the map $\Gamma\colon \mathcal{S}_V^d \to \mathcal{S}_{V^*}$ by

$$\Gamma(\phi) = \begin{bmatrix} 2aI & f \\ g^* & 2bI \end{bmatrix}.$$

It is easy to see that $\Gamma(\phi) \in \mathcal{S}_{V^*}$ for every $\phi \in \mathcal{S}_V^d$ and Γ is a surjective $\mathbb{C}$-linear map. To see that it is self-adjoint, note that

$$\Gamma(\phi^*) = \begin{bmatrix} 2\overline{a}I & \overline{g} \\ \overline{f} & 2\overline{b}I \end{bmatrix} = \begin{bmatrix} 2\overline{a}I & g^* \\ f^* & 2\overline{b}I \end{bmatrix} = \Gamma(\phi)^*,$$

so Γ is a vector space $*$-isomorphism.

The next theorem shows that Γ is a completely positive map. In order to obtain the result, we recall the numerical radius of an operator. Given $T \in B(\mathcal{H})$, the numerical radius of T, denoted by $\omega(T)$, is $\omega(T) = \sup\{|\langle Tx, x\rangle| \colon x \in \mathcal{H}, ||x|| \leq 1\}$. The next lemma about $\omega(T)$ will be used, and we include the proof for completeness. For a deeper study of numerical radius and matrix range, we refer the reader to [25, 26, 33].

Lemma 5.8. *Let $T \in B(\mathcal{H})$, $\omega(T) \leq 1$ if and only if for every unimodular λ, $2 + \lambda T + (\lambda T)^* \geq 0$. Moreover, $\omega(T) \leq ||T|| \leq 2\omega(T)$.*

Proof. It is easy to see that $2 + \lambda T + (\lambda T)^* \geq 0$ if and only if $1 + \text{Re } \lambda\langle Tx, x\rangle \geq 0$ for each $x \in \mathcal{H}$. For each $x \in \mathcal{H}$, choose a unimodular λ such that $\lambda\langle Tx, x\rangle = -|\langle Tx, x\rangle|$. Then $|\langle Tx, x\rangle| = -\lambda\langle Tx, x\rangle = -\text{Re } \lambda\langle Tx, x\rangle \leq 1$. Conversely, if $\omega(T) \leq 1$, for every unimodular λ and $x \in \mathcal{H}$, $-\text{Re } \lambda\langle Tx, x\rangle \leq |\lambda\langle Tx, x\rangle| \leq |\langle Tx, x\rangle| \leq \omega(T) \leq 1$. Thus, $2 + \lambda T + (\lambda T)^* \geq 0$. $\square$

Proposition 5.9. The map Γ is unital and completely positive, but not a complete order isomorphism.

Proof. It is easy to see that Γ is unital. Suppose $\Phi = [\phi_{ij}] \in M_n(\mathcal{S}_V^d)^+$, and $[T_{ij}] = \Gamma^{(n)}(\Phi) \in M_n(\mathcal{S}_{V^*})$. Write $A = [a_{ij}] = [\Phi(I \otimes E_{11})]$, $B = [b_{ij}] = [\Phi(I \otimes E_{22})]$, and $F(X) = [f_{ij}(X)] = [\Phi(X \otimes E_{12})]$, so A and B are positive scalar matrices. By positivity of Φ, we deduce that

$$T_{ij} = \begin{bmatrix} 2a_{ij}I & f_{ij} \\ f_{ji}^* & 2b_{ij}I \end{bmatrix}.$$

By the canonical shuffle, $[T_{ij}]$ is positive if and only if $\left[\begin{smallmatrix} 2A & F \\ F^* & 2B \end{smallmatrix}\right]$ is positive in $M_2(M_n(\mathcal{A}'))$, which by Lemma 5.3, is equivalent to

$$||A^{-1/2}FB^{-1/2}|| \leq 2. \tag{$\dagger\dagger$}$$

Recall that the norm on F in $M_n(\mathcal{A}')$ is the cb-norm of $F\colon V \to M_n$, so we will be done if we show $||F||_{cb} \le 2||A||^{1/2}||B||^{1/2}$.

To this end, recall that $||F||_{cb} = ||F^{(n)}||$. Let $X = [X_{kl}] \in M_n(V)$ with $||X|| = 1$. Note that $\left[\begin{smallmatrix} I_n & X \\ X^* & I_n \end{smallmatrix}\right] \in M_2(M_n(\mathcal{A}))$ is positive. By left and right multiplying this matrix by $B^{1/2} \oplus A^{1/2}$, one sees that

$$\begin{bmatrix} B & B^{1/2}XA^{1/2} \\ A^{1/2}X^*B^{1/2} & A \end{bmatrix} \in M_2(M_n(\mathcal{A}))^+.$$

Let $Q = [Q_{kl}] \in M_n(\mathcal{S}_V)^+$, such that Q is unitarily equivalent to the above matrix via the canonical shuffle. Since $\Phi\colon \mathcal{S}_V \to M_n$ is completely positive, $[\Phi(Q_{kl})] \in M_{n^2}^+$, but this matrix is precisely

$$A \otimes B + B \otimes A + F^{(n)}(B^{1/2}XA^{1/2}) + F^{(n)}(B^{1/2}XA^{1/2})^*.$$

Write $\kappa = F^{(n)}(B^{1/2}XA^{1/2})$, which equals to $(B^{1/2} \otimes I_n)F^{(n)}(X)(A^{1/2} \otimes I_n)$. Replace X by $e^{i\theta}X$, we deduce that

$$2||A||||B||(I_{n^2}) + e^{i\theta}\kappa + e^{-i\theta}\kappa^* \ge 0,$$

for all θ.

By the last lemma, $\omega(\kappa) \le ||A||||B||$, so $||\kappa|| \le 2||A||||B||$. Hence,

$$\begin{aligned} ||F^{(n)}(X)||_{M_{n^2}} &= ||(B^{-1/2} \otimes I_n)\kappa(A^{-1/2} \otimes I_n)|| \\ &\le ||B||^{-1/2}(2||A||||B||)||A||^{-1/2} \\ &= 2||A||^{1/2}||B||^{1/2}, \end{aligned}$$

for all $X \in M_n(V)$, where $||X|| = 1$. Therefore, $||F||_{cb} \leq 2||A||^{1/2}||B||^{1/2}$ and (††) is verified. Consequently, $[T_{ij}]$ is positive in $M_n(\mathcal{S}_{V^*})$, and Γ is a unital completely positive map.

Now if Γ was a complete order isomorphism, then for any $f\colon V \to \mathbb{C}$ such that $||f|| = 1$, the map $\phi\colon \mathcal{S}_V \to \mathbb{C}$ such that $\Gamma(\phi) = \left[\begin{smallmatrix} I & f \\ f^* & I \end{smallmatrix}\right]$ would be a positive linear functional. But it is easily checked that

$$\phi\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}\right) = \frac{\lambda + \mu}{2} + f(x) + \overline{f(y)}.$$

So ϕ of the identity is 1, and hence, ϕ is a state. But by Theorem 5.5, $||f|| \leq \frac{1}{2}$, which is a contradiction. Hence, Γ is not a complete order isomorphism. $\square$

5.3 An Operator System Quotient of S_V

Let $\mathcal{S}$ be an operator system with a kernel $\mathcal{J}$. In [18], it is shown that the quotient $\mathcal{S}/\mathcal{J}$ has two natural operator space structures induced from $\mathcal{S}$: one from the operator space quotient and one from the operator system quotient. We denote each structure by $(\mathcal{S}/\mathcal{J})_{ops}$ and $(\mathcal{S}/\mathcal{J})_{opsys}$ and their norms by $||\cdot||_{ops}^{(n)}$ and $||\cdot||_{opsys}^{(n)}$, respectively. More precisely, by Proposition 4.1 in [18],

$$||([x_{ij} + \mathcal{J})||_{osp}^{(n)} = \sup\{||[\phi(x_{ij})]||\},$$
$$||([x_{ij} + \mathcal{J})||_{opsys}^{(n)} = \sup\{||[\psi(x_{ij})]||\},$$

where $\phi, \psi\colon \mathcal{S} \to B(\mathcal{H})$ vanishes on $\mathcal{J}$, ϕ is completely contractive, and ψ is unital completely positive. It is known that the identity map $(\mathcal{S}/\mathcal{J})_{ops} \to (\mathcal{S}/\mathcal{J})_{opsys}$ is a complete contraction; that is $||\cdot||^{(n)}_{opsys} \leq ||\cdot||^{(n)}_{ops}$, for each n.

It is then natural to ask when the two structures are completely bounded-isomorphic (or completely norm-equivalent). In [18], the question is answered by decomposability of completely positive maps with respect to the kernel $\mathcal{J}$.

Definition 5.10. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\mathcal{J}$ be a kernel of $\mathcal{S}$. A completely bounded map $\phi\colon \mathcal{S} \to \mathcal{T}$ with $\phi(\mathcal{J}) = \{0\}$ is called $\mathcal{J}$-decomposable if there exist completely positive maps $\phi_i\colon \mathcal{S} \to \mathcal{T}$, with $\phi_i(\mathcal{J}) = \{0\}$, $i = 1, 2, 3, 4$, such that $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$.

It turns out that such ϕ is $\mathcal{J}$-decomposable if and only if there are completely positive maps $\psi_1, \psi_2\colon \mathcal{S} \to \mathcal{T}$ vanishing on $\mathcal{J}$ such that the map $\Phi\colon \mathcal{S} \to M_2(\mathcal{T})$ given by

$$\Phi(x) = \begin{bmatrix} \psi_1(x) & \phi(x) \\ \phi(x)^* & \psi_2(x) \end{bmatrix}$$

is completely positive. This observation leads to the following definition:

$$||\phi||_{\mathcal{J}-dec} = \inf \max\{||\psi_1||, ||\psi_2||\},$$

over all such ψ_i as above.

By Theorem 4.10 in [18], $\mathcal{J}$-decomposability characterizes norm equivalence of the two operator space quotient structures.

Theorem 5.11. *Let $\mathcal{S}$ be an operator system and $\mathcal{J}$ be a kernel in $\mathcal{S}$. Then the following are equivalent:*

1. *For every Hilbert space $\mathcal{H}$, every completely bounded map from $\mathcal{S}$ into $\mathcal{B}(\mathcal{H})$ that vanishes on $\mathcal{J}$ is $\mathcal{J}$-decomposable.*

2. *There exists a constant C such that for all n and all $[x_{ij}] \in M_n(\mathcal{S})$, we have that $||[x_{ij} + \mathcal{J}]||^{(n)}_{osp} \leq C||[x_{ij} + \mathcal{J}||^{(n)}_{opsys}$.*

Moreover, in these cases the least such constant C is equal to the least constant satisfying $||\phi||_{\mathcal{J}-dec} \leq C||\phi||_{cb}$, for all completely bounded maps $\phi\colon \mathcal{S} \to \mathcal{B}(\mathcal{H})$ vanishing on $\mathcal{J}$.

Nevertheless, checking the first condition of this theorem is rather difficult. For example, when $\mathcal{S} = \mathcal{S}_V$, finding $||\phi||_{\mathcal{J}-dec}$ for just one completely bounded map ϕ on $\mathcal{S}_V$ amounts to checking $\Phi(x)$, where each x is in fact a 2×2 operator matrix.

In the next definition, we establish an easier notion that only depends on $\mathcal{S}$ and $\mathcal{T}$, and it is sufficient to conclude complete norm equivalence of the two operator space structures. The following notion can also be found in the work of Ortiz and Paulsen [24, Section 4].

Definition 5.12. Let $\mathcal{J}$ be a kernel of an operator system $\mathcal{S}$. We define

$$\Theta_n(\mathcal{J}) = \sup\{||e_n + [y_{ij}]||\colon y_{ij} \in \mathcal{J},\ e_n + [y_{ij}] \in M_n^+(\mathcal{S})\}.$$

We also define $\Theta_{cb}(\mathcal{J}) = \sup_n \Theta_n(\mathcal{J})$.

Proposition 5.13. If $\Theta_{cb}(\mathcal{J}) < \infty$, then the two quotient norms are completely equivalent.

Proof. Consider the identity map $\Gamma\colon (\mathcal{S}/\mathcal{J})_{opsys} \to (\mathcal{S}/\mathcal{J})_{ops}$ and suppose $\Theta_{cb}(\mathcal{J}) < \infty$. We claim that $||\Gamma||_{cb} \leq \Theta_{cb}(\mathcal{J})$. Let $[x_{ij} + \mathcal{J}] \in M_n((\mathcal{S}/\mathcal{J})_{opsys})$ with norm 1, so that

$$\begin{bmatrix} e_n & [x_{ij}] \\ [x_{ij}]^* & e_n \end{bmatrix} + M_{2n}(\mathcal{J}) \in M_{2n}(\mathcal{S}/\mathcal{J})^+.$$

By the lifting property, for each $\varepsilon > 0$, there exist $a_{ij}, b_{ij}, c_{ij} \in \mathcal{J}$ (dependent on ε) such that

$$\begin{bmatrix} (1+\varepsilon)e_n + [a_{ij}] & [x_{ij} + b_{ij}] \\ [x_{ij} + b_{ij}]^* & (1+\varepsilon)e_n + [c_{ij}] \end{bmatrix} \in M_{2n}(\mathcal{S})^+.$$

By Lemma 5.3,

$$||[x_{ij}] + [b_{ij}]||^2 \leq ||(1+\varepsilon)e_n + [a_{ij}]|| \cdot ||(1+\varepsilon)e_n + [c_{ij}]|| \leq (1+\varepsilon)^2 \Theta_{cb}(\mathcal{J})^2,$$

by the definitions of the operator space quotient and of $\Theta_{cb}(\mathcal{J})$. Since $\varepsilon > 0$ is arbitrary,

$$||[x_{ij} + \mathcal{J}]||_{ops}^{(n)} \leq \Theta_{cb}(\mathcal{J}).$$

Therefore, $||\Gamma||_{cb} \leq \Theta_{cb}(\mathcal{J})$ and the two quotient norms are equivalent. $\square$

We are now ready to examine a natural quotient of $\mathcal{S}_V$.

Proposition 5.14. The subspace

$$\mathcal{J} = \left\{ \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix} \in \mathcal{S}_V : \lambda \in \mathbb{C} \right\},$$

is a kernel of $\mathcal{S}_V$.

Proof. Let $\mathcal{I}$ be the set of linear functional f on V with $||f|| \leq \frac{1}{2}$. Define the map $\Phi\colon \mathcal{S}_V \to \oplus_{f\in\mathcal{I}}\mathbb{C} = l^\infty(\mathcal{I})$ by

$$\Phi\left(\begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix}\right) = \left(\frac{1}{2}\lambda + \frac{1}{2}\mu + f(X) + \overline{f}(Y)\right)_{f\in\mathcal{I}}.$$

Since $l^\infty(\mathcal{I})$ is an abelian C*-algebra and for each $f \in \mathcal{I}$, $\frac{1}{2}(\lambda + \mu) + f(X) + \overline{f(Y)}$ defines a state on $\mathcal{S}_V$ by Theorem 5.5, the map Φ is unital completely positive.

We now claim that $\mathcal{J} = \ker(\Phi)$. It is obvious that $\mathcal{J} \subset \ker(\Phi)$. Suppose $\left[\begin{smallmatrix} \lambda I & X \\ Y^* & \mu I \end{smallmatrix}\right] \in \ker(\Phi)$ but not in $\mathcal{J}$. If $\mu \neq -\lambda$, take f to be the zero functional and the f-th entry of $\Phi\left(\left[\begin{smallmatrix} \lambda I & X \\ Y^* & \mu I \end{smallmatrix}\right]\right)$ is non-zero, which is a contradiction. Hence, μ must be $-\lambda$.

Now suppose $X = Y \neq 0$ and without loss of generality, let $||X|| = 1$. By the Hanh-Banach Theorem, there is a linear functional f on V with $||f|| = 1$ and $f(X) = 1$. Take $g = \frac{1}{2}f \in \mathcal{I}$, so that the g-th entry is non-zero, again a contradiction. Finally, if $X \neq Y$ and both are not 0, then the matrix

$$\begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix} + \begin{bmatrix} \lambda I & X \\ Y^* & \mu I \end{bmatrix}^*$$

is selfadjoint and is in $\ker(\Phi)$. Apply the previous step and we have a contradiction. Therefore, $\ker(\Phi) = \mathcal{J}$ and $\mathcal{J}$ is a kernel of $\mathcal{S}_V$. □

By Proposition 1.3, $\mathcal{S}_V/\mathcal{J}$ is a quotient operator system. By duality of quotient maps, we conclude that its dual is again an operator system.

Corollary 5.15. The matrix order space $(\mathcal{S}_V/\mathcal{J})^d$ is an operator subsystem of $\mathcal{S}_V^d$.

Proof. Since the quotient map $q \colon \mathcal{S}_V \to \mathcal{S}_V/\mathcal{J}$ is a complete quotient, its dual map $q^d \colon (\mathcal{S}_V/\mathcal{J})^d \to \mathcal{S}_V^d$ is a complete order embedding between two matrix ordered spaces. Recall that $\mathcal{S}_V^d$ is an operator system with unit $\frac{1}{2}tr$, so it remains to choose a faithful state on $\mathcal{S}_V/\mathcal{J}$ that serves as an Archimedean order unit on its dual space. Let $\overline{tr}(X + \mathcal{J}) = \frac{1}{2}tr(X)$ on $\mathcal{S}_V/\mathcal{J}$, it is elementary to check that $\overline{tr}$ is an Archimedean matrix order unit for $(\mathcal{S}_V/\mathcal{J})^d$ and $q^d(\overline{tr}) = \frac{1}{2}tr$. □

We remark that Proposition 5.14 can be proved using Lemma 2.3 in [15] in a more general setting. By Kavruk's result, $\mathcal{J}$ is **completely proximinal**; that is, the matricial order structure of $\mathcal{S}_V/\mathcal{J}$ is given by

$$M_n(\mathcal{S}_V/\mathcal{J})^+ = \{[x_{ij} + \mathcal{J}] \colon [x_{ij}] \in M_n(\mathcal{S}_V)^+\},$$

without the need of Archimedeanization.

In the next result, we use $\Theta_{cb}(\mathcal{J})$ to show that the two natural operator space structures on $\mathcal{S}_V/\mathcal{J}$ are completely bounded-isomorphic.

Proposition 5.16. Let $\mathcal{S}_V$ and $\mathcal{J}$ be as above, then $(\mathcal{S}_V/\mathcal{J})_{ops}$ and $(\mathcal{S}_V/\mathcal{J})_{opsys}$ are completely bounded-isomorphic.

Proof. Since $\mathcal{J} = \mathbb{C}(1 \otimes E_{11} - 1 \otimes E_{22})$, for each $n \in \mathbb{N}$, let $\varepsilon > 0$ and $y_{ij} \in \mathcal{J}$ so that

$$(1+\varepsilon)I_n \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + [y_{ij}] \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \geq 0.$$

Then

$$\begin{bmatrix} (1+\varepsilon)I_n + [y_{ij}] & 0 \\ 0 & (1+\varepsilon)I_n - [y_{ij}] \end{bmatrix} \in M_n(\mathcal{S}_V)^+ \subset M_{2n}(\mathcal{A})^+.$$

Hence, the matrix $[y_{ij}]$ is a contraction, and $||I_n + [y_{ij}]|| \leq 2$. By taking the supremum over all such y_{ij}'s and all n, we deduce that $\Theta_{cb}(\mathcal{J}) \leq 2$. By Proposition 5.13, $(\mathcal{S}_V/\mathcal{J})_{ops}$ and $(\mathcal{S}_V/\mathcal{J})_{opsys}$ are completely bounded-isomorphic. $\square$

From the above proof, we see that on $\mathcal{S}_V/\mathcal{J}$, $||\cdot||^{(n)}_{opsys} \leq ||\cdot||^{(n)}_{ops} \leq 2||\cdot||^{(n)}_{opsys}$ for each n. By Theorem 5.11, every completely bounded map $\phi \colon \mathcal{S}_V \to \mathcal{T}$ vanishing on $\mathcal{J}$ is $\mathcal{J}$-decomposable. Finally, we end this chapter with some interesting open questions concerning on $\mathcal{S}_V$.

Question 5.17. In [3], it is shown that if $\mathcal{A}'$ is the dual Banach space of a nontrivial C*-algebra $\mathcal{A}$ with its usual cone, then there does not exist any isometric positive map from $\mathcal{A}'$ into another C*-algebra. Hence, $\mathcal{A}'$ cannot be an ordered operator space. Indeed one can take $\mathcal{A} = l_2^\infty$. In this regard, what can we conclude about $\mathcal{S}_\mathcal{A}$, $\mathcal{S}_{\mathcal{A}^*}$, and $\mathcal{A}'$?

Question 5.18. In Proposition 5.9, we saw that the natural map Γ between $\mathcal{S}_V^d$ and $\mathcal{S}_{V^*}$ is not a unital complete order isomorphism. Is there any other natural map between these two operator systems that is a unital complete order isomorphism?

Question 5.19. Is the quotient operator system $\mathcal{S}_V/\mathcal{J}$ self-dual? If so, is its cb-condition number equal to 2? How is it related to $\Theta_{cb}(\mathcal{J})$?

Question 5.20. The embedding $V \to \mathcal{S}_V$ by $T \mapsto T \otimes E_{12}$ is a complete isometric inclusion. What about $V \mapsto \mathcal{S}_V/\mathcal{J}$ by $T \mapsto T \otimes E_{12} + \mathcal{J}$, or $T \mapsto T \otimes E_{12} + T^* \otimes E_{21} + \mathcal{J}$? Do these maps preserve the numerical radius $\omega(T)$, instead of $||T||$?

Bibliography

[1] W. B. Arveson, *Subalgebras of C*-algebras*, Acta Math. **123** (1969), 141–224.

[2] D. P. Blecher and B. L. Duncan, *Nuclearity-related properties for nonselfadjoint algebras*, J. Operator Theory **65** (2011) 47–70.

[3] D. P. Blecher, K. Kirkpatrick, M. Neal, and W. Werner, *Ordered involutive operator spaces*, Positivity **11** (2007), no. 3, 497–510.

[4] D. P. Blecher and C. Le Merdy *Operator algebras and their modules — An operator space approach*, Oxford Univ. Pres, 2004.

[5] D. P. Blecher and V. I. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. **99** (1991) 262–292.

[6] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.

[7] M. D. Choi and E. G. Effros, *Nuclear C*-algebras and the approximation property*, Amer. J. Math. **100** (1978), 61–79.

[8] D. Farenick, A. S. Kavruk, V. I. Paulsen, and I. G. Todorov, *Operator systems from discrete groups*, Communications in Mathematical Physics **329** (2014), no. 1, 207–238.

[9] D. Farenick, A. S. Kavruk, V. I. Paulsen, and I. G. Todorov, *Characterisations of the weak expectation property*, arXiv:1307.1055 (2013).

[10] D. Farenick and V. I. Paulsen, *Operator system quotients of matrix algebras and their tensor products*, arXiv:1101.0790 (2011).

[11] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Séminaire Bourbaki **2** (1955), 193–200.

[12] V. P. Gupta and P. Luthra, *Operator system nuclearity via C*-envelopes*, arXiv:1408.4312 (2015).

[13] Uffe Haagerup, *Injectivity and decomposition of completely bounded maps*, Springer, 1985.

[14] K. H. Han and V. I. Paulsen, *An approximation theorem for nuclear operator systems*, J. Func. Anal. **261** (2011), no. 4, 999–1009.

[15] A. S. Kavruk, *Nuclearity related properties in operator systems*, J. Operator Theory **71** (2014), 95–156.

[16] A. S. Kavruk, *On a non-commutative analogue of a classical result of Namioka and Phelps*, J. Funct. Anal. **269** (2015), no. 10, 3282–3303.

[17] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Tensor products of operator systems*, J. Funct. Anal. **261** (2011), no. 2, 267–299.

[18] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Quotients, exactness, and nuclearity in the operator system category*, Advances in Mathematics **235** (2013), 321–360.

[19] E. Kirchberg, *C*-nuclearity implies CPAP*, Math. Nachr. **76** (1977), 203–212.

[20] E. Kirchberg and S. Wassermann, *C*-algebras generated by operator systems*, J. Funct. Anal. **155** (1998), no. 2, 324–351.

[21] C. Lance, *On nuclear C*-algebras*, J. Funct. Anal. **12** (1973), no. 2, 157–176.

[22] W. H. Ng, *Two characterizations of the maximal tensor product of operator system*, arXiv:1503.07097 (2015).

[23] W. H. Ng and V. I. Paulsen *The SOH operator system*, J. Operator Theory, to appear.

[24] C. M. Ortiz and V. I. Paulsen, *Quantum Graph Homomorphisms via Operator Systems*, arXiv:1505.00483 (2015).

[25] V. I. Paulsen, *Preservation of essential matrix ranges by compact perturbations*, J. Operator Theory **8** (1982), 299–317.

[26] V. I. Paulsen, *Completely bounded maps and operator algebras*, vol. 78, Cambridge Univ. Press, 2002.

[27] V. I. Paulsen and M. Tomforde, *Vector spaces with an order unit*, IMUJ, **58** (2009), no. 3 1319–1359.

[28] V. I. Paulsen, I. G. Todorov, and M. Tomforde, *Operator system structures on ordered spaces*, Proc. of the London Math. Soc., **102** (2010), no. 1, 25–49.

[29] G. Pisier, *The operator Hilbert space OH and type III von Neumann algebras*, Bull. of the London Math. Soc. **36** (2004), no. 4, 455–459.

[30] G. Pisier, *The operator Hilbert space OH, complex interpolation, and tensor norms*, vol. 122, American Mathematical Soc., 1996.

[31] G. Pisier, *Introduction to operator space theory*, vol. 294, Cambridge Univ. Press, 2003.

[32] V. Rajpal, A. Kumar, and T. Itoh, *Schur tensor product of operator spaces*, arXiv:1308.4538 (2013).

[33] R. R. Smith and J. D. Ward, *Matrix ranges for hilbert space operators*, Amer. J. Math. (1980), 1031–1081.

[34] Gerd Wittstock, *Ein operatorwertiger Hahn-Banach satz*, J. Funct. Anal. **40** (1981), no. 2, 127–150.

[35] C. Zhang, *Representation and geometry of operator spaces*, Ph.D. thesis, University of Houston, 1995.