AN APPROACH TO THE REDUCTION OF MULTIVARIABLE SYSTEMS WITH VARIOUS NUMBERS OF INPUTS AND OUTPUTS

A Thesis

Presented to

the Faculty of the Department of Electrical Engineering

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Electrical Engineering

---

by
Hung Ching Lue
August 1975

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ABSTRACT

The analysis and synthesis of high-order systems are computationally difficult and cumbersome. Accordingly, there is a need for obtaining reduced models for the high-order system so that an analogue or digital simulation of the system is possible. An algebraic method is proposed in the frequency domain to obtain the reduced models of single-variable systems as well as multivariable systems. The method of matrix-continued fraction and the mixed method, which utilizes both the dominant-eigenvalue concept and matrix-continued fraction approach, are extended to obtain the reduced models. The reduced model is always stable and it retains the dominant performance of the original system. A complete computer-oriented algorithm is established for the simplification.

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CHAPTER I

INTRODUCTION

In general, the practical systems are highly dimensional, heavily coupled, and have a multiplicity of inputs and/or outputs. Exact analysis and synthesis of such a high-order multivariable system are tedious and costly processes. It is always desirable to research for a reduced model, so that an analogue or digital simulation of the system is possible. The technique of system reduction has been recently investigated by numerous authors.[1-8] The principle of model reduction is to discard the unimportant terms and retain the significant terms of interest. It has been recognized[9] that the most powerful method for system reduction of a high-order transfer function was developed by Chen and Shieh.[2,4] This method has been extended to simplify a high-degree transfer-function matrix, using the matrix-continued fraction as a basis.

The matrix-continued fraction can be summarized: the denominator and numerator polynomials in the transfer-function matrix are arranged in the ascending order in powers of S. After expanding into matrix-continued fraction, it has been shown[6] that the matrix quotients in the expansion are in the order of decreasing significance as far

as their contribution to the system response is concerned. Thus, we can truncate the least significant matrix quotients and obtain the desired reduced model; however, this method has a disadvantage in that the reduced model may be unstable even though the original system is stable.

Another popular technique for handling the reduction problem is the dominant-pole approach proposed by Davison[1] and Chidambara.[3] This method is based on the concept that the poles of the original system that are far away from the jw-axis in the S-plane can be neglected. Thus, a reduced model can be constructed by retaining the dominant poles of the original system. The most important feature of this approach is that the reduced model is always stable and dominant performance of the original system may be maintained. However, this method involves complicated linear transformation, steady-state value matching, and matrix diagonalization. The computational procedures are very cumbersome when the order of a multivariable system is high.

Moreover, the existing methods[5-7] in the frequency domain only deal with multivariable systems with an equal number of inputs and outputs and the transfer-function matrix has no ill-conditional numerical elements. In this research, the matrix-continued fraction and the mixed method are extended to obtain reduced models of the general single-variable systems, as well as multivariable systems. The proposed methods can be applied to the approximation of

multivariable systems with various numbers of inputs and outputs; a technique is also established for dealing with ill-conditioned cases. The procedure proposed in this research is simple in theory and flexible in practice. The entire process can be performed by a digital computer.

CHAPTER II

SYSTEM REDUCTION WITH EQUAL NUMBERS OF INPUTS AND OUTPUTS

To demonstrate the principles and procedures of system reduction, we will review and extend some of the existing methods as follows:

Section 1. Single-Input Single-Output System

a) The Second Cauer Form

In this research, system reduction is performed in the frequency domain. Obviously, the expression of the control system is the transfer function in the S-domain. Consider the following single-variable system:

$$T(S) = \frac{A_{21} + A_{22}S + A_{23}S^2 + \ldots + A_{2,n}S^{n-1}}{A_{11} + A_{12}S + A_{13}S^2 + \ldots + A_{1,n+1}S^n} \tag{1}$$

where $A_{i,j}$ are constants and $A_{1,n+1=1}$. This equation can be expanded into the second Cauer form as follows[4]:

$$T(S) = \cfrac{1}{\cfrac{A_{11}}{A_{21}} + \cfrac{\cfrac{A_{21}A_{12} - A_{11}A_{22}}{A_{21}}S + \cfrac{A_{21}A_{13} - A_{11}A_{23}}{A_{21}}S^2 + \ldots.}{A_{21} + A_{22}S + A_{23}S^2 + \ldots. + A_{21n}S^{n-1}}} \tag{2}$$

where

$$A_{31} = \frac{A_{21}A_{12} - A_{11}A_{22}}{A_{21}}$$

$$A_{32} = \frac{A_{21}A_{13} - A_{11}A_{23}}{A_{21}} \tag{3}$$

Then Eq. (2) becomes

$$T(S) = \cfrac{1}{\cfrac{A_{11}}{A_{21}} + \cfrac{A_{31}S + A_{32}S^2 + A_{33}S^3 + \ldots}{A_{21}S + A_{22}S + A_{23}S^2 + \ldots}} \tag{4}$$

Performing the division again,

$$T(S) = \cfrac{1}{\cfrac{A_{11}}{A_{21}} + \cfrac{S}{\cfrac{A_{21}}{A_{31}} + \cfrac{\cfrac{A_{22}A_{31} - A_{32}A_{21}}{A_{31}}S + \ldots}{A_{31} + A_{32}S + \ldots}}} \tag{5}$$

where

$$A_{41} = \frac{A_{22}A_{31} - A_{32}A_{21}}{A_{31}} \tag{6}$$

Eq. (5) is written

$$T(S) = \cfrac{1}{\cfrac{A_{11}}{A_{21}} + \cfrac{1}{\cfrac{A_{21}}{A_{31}}S + \cfrac{1}{\cfrac{A_{31}}{A_{41}} + \cfrac{1}{\ddots}}}} \tag{7}$$

If we set

$$h_1 = \frac{A_{11}}{A_{21}} \quad ,$$

$$h_2 = \frac{A_{21}}{A_{31}} \quad ,$$

$$h_3 = \frac{A_{31}}{A_{41}} \quad , \text{ etc.} \tag{8}$$

Then the continued-fraction expansion of the second Cauer form is

$$T(S) = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{S} + \cfrac{1}{h_3 + \cfrac{1}{\ddots}}}} \tag{9}$$

The general formula to evaluate the scaler quotients in Eq. (9) can be obtained by the following Routh algorithm.[10]

$$\begin{array}{lll} & A_{11} \quad A_{12} \quad A_{13} \quad A_{14} \quad \cdots \\ h_1 = \dfrac{A_{11}}{A_{21}} & < \\ & A_{21} \quad A_{22} \quad A_{23} \quad \cdots \\ h_2 = \dfrac{A_{21}}{A_{31}} & < \\ & A_{31} \quad A_{32} \quad \cdots \\ h_3 = \dfrac{A_{31}}{A_{41}} & < \\ & A_{41} \quad \cdots \\ \cdots & \cdots \end{array} \tag{10}$$

where $A_{i,j} = A_{i-2,j+1} - h_{i-2} A_{i-1,j+1}$; $i=3,4,\ldots,2n+1$;

$j=1,2,\ldots,n$

$$h_i = \frac{A_{i,1}}{A_{i+1,1}} \quad ; \quad i=1,2,\ldots,2k \text{ and } k \leq n$$

$$\det (A_{i+1,1}) \neq 0$$

If the quotients are given and the corresponding transfer function is required, then this process is called continued-fraction inversion. Considering that four scaler quotients h_i, $i=1,\ldots,4$ are given, the continued fraction inversion of Eq. (9) is:

$$T(S) = \frac{h_2h_3h_4 + \left(h_2 + h_4\right)S}{h_ih_2h_3h_4 + \left(h_ih_2 + h_1h_4 + h_3h_4\right)S + S^2} \tag{11}$$

In general, the transfer function of Eq. (1) can be obtained by the following inverse process of the Routh algorithm[10]

$$A_{2n+1,1} = 1$$

$$A_{i,1} = h_i A_{i+1,1}; \; i=2n,2n-1,\ldots,2,1$$

$$A_{j-2,\ell+1} = A_{j,\ell} + h_{j-2} A_{j-1,\ell+1}; \; j=2n+1,2n,\ldots,3; \quad \ell=1,2,\ldots,n \tag{12}$$

It is observed that the first several quotients are dominant ones. This can be verified by again considering Eq. (11)

$$T(S) = \frac{C(S)}{R(S)} = \frac{\left(h_2 + h_4\right)S + h_2h_3h_4}{S^2 + \left(h_1h_2 + h_ih_4 + h_3h_4\right)S + h_ih_2h_3h_4} \tag{13}$$

Applying the final value theorem to Eq. (13) and allowing $R(S) = \frac{1}{S}$, it is found that

$$C(t)\Big|_{t\to\infty} = \frac{1}{h_1} \tag{14a}$$

Similarly, applying the initial value theorem and allowing $R(S) = 1$,

$$C(t)\Big|_{t\to 0} = h_2 + h_4 \tag{14b}$$

The results obtained in Eqs. (14a) and (14b) imply that the quotient h_1 dominates the final or steady-state value of the behavior of the system. In other words, the second Cauer form influences very heavily the steady-state part of the system response. It should be noted that the most dominant term is h_1 and the second influence term is h_2. When the quotients in the continued fraction are well-distributed and lower and lower in position, they are less and less important as far as the influence to the performance of the system is concerned. This observation is the general basis for the simplification technique developed in this research for both the continued-fraction method and the mixed method.

b) Mixed Method

It is well known that the reduced model obtained by applying Eqs. (9) and (13) may be unstable, even though the original system is stable. Hence, the approximation by continued fraction does not necessarily yield a stable model. To overcome this deficiency, the mixed method[7] is extended

for the reduction of both single-variable and multivariable systems. The approach involved in the mixed method follows. The denominator of Eq. (1) can be factorized as

$$\Delta(S) = (S - \lambda_1)(S - \lambda_2) \ldots (S - \lambda_n) \tag{15}$$

If p dominant poles are chosen as the dominant eigenvalues of the reduced model, then the new denominator polynomial of a reduced model is

$$\Delta p(S) = (S - \lambda_1)(S - \lambda_2) \ldots (S - \lambda_P)$$

$$= \sum_{j=1}^{p+1} d_j \, S^{j-1}, \; d_{p+1} = 1$$

$$b_{1,j} = d_j \quad , \quad j=1,2,\ldots,p \tag{16}$$

and p is the degree of reduced model and $b_{1,\,p+1} = 1$.

The scaler gquotients h_i, $i=1,2,\ldots\ell$, can be evaluated by the algorithm shown in Eq. (10). When the coefficients $b_{i,j}$ of the reduced model in Eq. (16) and the dominant quotients h_i in Eq. (10) are found, the coefficients $b_{2,j}$ of the numerator polynomial can be evaluated by the following new Routh algorithm[10],

$$b_{i+1,1} = \frac{b_{i,1}}{h_i} \quad , \; i=1,2,\ldots,p \text{ and } p \leq n$$

$$b_{i+1,j+1} = \frac{b_{i,j+1} - b_{i+2,j}}{h_i} \quad , \; i=1,2,\ldots,p-j; \; j=1,2,\ldots,p-1$$

$$b_{1,p+1} = 1 \tag{17}$$

There is an alternative Routh algorithm that yields identical results.[4] Considering Eq. (10) as follows:

$$\begin{array}{cccc} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{12}-h_1b_{22} & b_{13}-h_1b_{23} & b_{14}-h_1b_{24} & \cdots \\ b_{22}-h_2\left(b_{13}-h_1b_{23}\right) & b_{23}-h_2\left(b_{14}-h_1b_{24}\right) & \cdots & \cdots \end{array} \tag{18}$$

it is seen that

$$h_1 = \frac{b_{11}}{b_{21}},$$

$$h_2 = \frac{b_{21}}{b_{12}-h_1b_{22}},$$

$$h_3 = \frac{b_{12}-h_1b_{22}}{b_{22}-h_2\left(b_{13}-h_1b_{23}\right)},$$

$$h_4 = \frac{b_{22}-h_2\left(b_{13}-h_1b_{23}\right)}{b_{13}-h_1b_{23}-h_3\left[b_{23}-h_2\left(b_{14}-h_1b_{24}\right)\right]} \tag{19}$$

A matrix expression for Eq. (19) can be formulated as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & h_2 & 0 & 0 & \cdots \\ 0 & 1 & h_2h_3 & 0 & \cdots \\ 0 & 0 & h_2+h_4 & h_2h_3h_4 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ . \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & 0 & 0 & 0 & \cdots \\ 1 & h_1h_2 & 0 & 0 & \cdots \\ 0 & h_1+h_3 & h_1h_2h_3 & 0 & \cdots \\ 0 & 1 & (h_1h_2+h_1h_4+h_3h_4) & h_1h_2h_3h_4 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \\ b_{24} \\ . \end{bmatrix} \qquad (20)$$

Eq. (20) can be factored in an alternate form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & h_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & h_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & h_p \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & h_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & h_{p-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_2 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ \vdots \\ b_{1,p} \end{bmatrix}$$

$$= \begin{bmatrix} h_1 & 0 & \cdots & 0 & 0 \\ 1 & h_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & h_p \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & h_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & h_{p-1} \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1 \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \\ \vdots \\ b_{2,p} \end{bmatrix}$$

(21)

where p is the degree of reduced model and $b_{1,\,p} = 1$.

Eq. (21) can be written in the form

$$H_e \vec{b}_1 = H_r \vec{b}_2 \tag{22a}$$

where H_e and H_r are both the product of the corresponding bidiagonal matrices. Both H_e and H_r are nonsingular, provided $h_i \neq 0$. If $A_{1,j}$ and $h_{j,j=1,2,\ldots,\,p}$ are given in Eq. (21), $A_{2,j}$ can be computed as

$$\vec{b}_2 = H\vec{b}_1 \tag{22b}$$

where $H = H_r^{-1} H_e$.

The method of continued fraction for the reduction of the single-variable systems has been well exploded by Chen and Shieh.[2] The following examples are used to illustrate the advantages of the mixed method.

<u>Example 1</u>

$$T(S) = \frac{S^6+848\cdot5246S^5+33147\cdot6926S^4+200543\cdot1225S^3 +398977\cdot6385^2+226104\cdot43475+23407\cdot25497}{6.675^7+299\cdot905S^6+8534\cdot0506S^5+63508\cdot04115^4 +233963\ 9526S^3+411730\ 9385S^2+227275\ 1388S +23407\cdot25497} \tag{23}$$

The eigenvalues of the characteristic equation, $\Delta(S)$, is

$$\begin{aligned}\Delta(S) = {} & (S+0\cdot13247484)(S+0\cdot70408738)(S+2\cdot49655724+j2\cdot69364548)\\ & (S+2\cdot49655724-j2\cdot69364548)(S+3\cdot01673794)\\ & (S+18\cdot05841064+j24\cdot46449279)(S+18\cdot05841064-j24\cdot46449279)\end{aligned} \tag{24}$$

Expanding the above transfer function according to Eq. (10), we have 14 quotients, the first eight being:

$$h_1 = 1.0$$

$$h_2 = 19.9942$$

$$h_3 = -0.0405282$$

$$h_4 = -15.6876$$

$$h_5 = 0.476296$$

$$h_6 = 0.282881$$

$$h_7 = -25.6977$$

$$h_8 = 1.31265 \tag{25}$$

If the method of continued fraction is applied and $h_{i,i=1,\ldots,8}$, are used. Therefore, Eq. (23) becomes

$$T_4(S) = \cfrac{1}{1.0 + \cfrac{1}{\cfrac{19.0042}{S} + \cfrac{1}{-0.0405282 + \cfrac{1}{\cfrac{-15.6876}{S} + \cfrac{1}{0.476296}}}}}$$

$$+ \cfrac{1}{\cfrac{0.282881}{S} + \cfrac{1}{-25.6977 + \cfrac{1}{\cfrac{1.31265}{S}}}} \tag{26}$$

converting Eq. (26) into a regular transfer function by applying Eq. (12), it is found that

$$T_4(S) = \frac{5\cdot902194S^3-140\cdot12299S^2-430\cdot98636S-57\cdot775278}{S^4-26\cdot498854S^3-165\cdot24465S^2-433\cdot87596S-57\cdot775278} \tag{27}$$

Obviously the reduced model is unstable, even though the original system is stable. The approximation by continued fraction does not necessarily give a stable model.

If the mixed method is applied and $h_{i,i=1,\ldots,4}$, are used, then the four dominant poles used are

$$\begin{aligned}\Delta_4(S) &= (S+0\cdot13247484)(S+0\cdot70408738)(S+2\cdot49655724 \\ &\quad +j2\cdot69364548)(S+2\cdot49655724-j2\cdot69364548) \\ &= S^4+5\cdot8296802S^3+17\cdot758865S^2+11\cdot749763S+1\cdot2581257\end{aligned} \tag{28}$$

To obtain the simplified transfer function, we can apply either Eq. (17) or Eq. (22). A fourth-order approximation of the original seventh-order system is found to be:

$$T_4(S) = \frac{4\cdot2523635S^3+18\cdot421032S^2+11\cdot686839S+1\cdot2581257}{S^4+5\cdot8296802S^3+17\cdot758864S^2+11\cdot759763S+1\cdot2581257} \tag{29}$$

The impulse responses of the original and approximated systems are shown in Fig. 1. As expected, there is a small error in the initial-state portion of the approximated response curve. With a unit step input, the response of the original system and the fourth-order simplified system of Eq. (29) is shown in Fig. 2.

<u>Example 2</u>

Consider the following system

$$T(S) = \frac{1464\cdot786701S^3+79582\cdot5474S^2+533760\cdot7473S+617497\cdot375}{\begin{aligned}&S^7+112\cdot04S^6+3755\cdot92S^5+39736\cdot62S^4+363650\cdot56S^3 \\ &+759894\cdot19S^2+683656\cdot25S+617497\cdot375\end{aligned}} \tag{30}$$

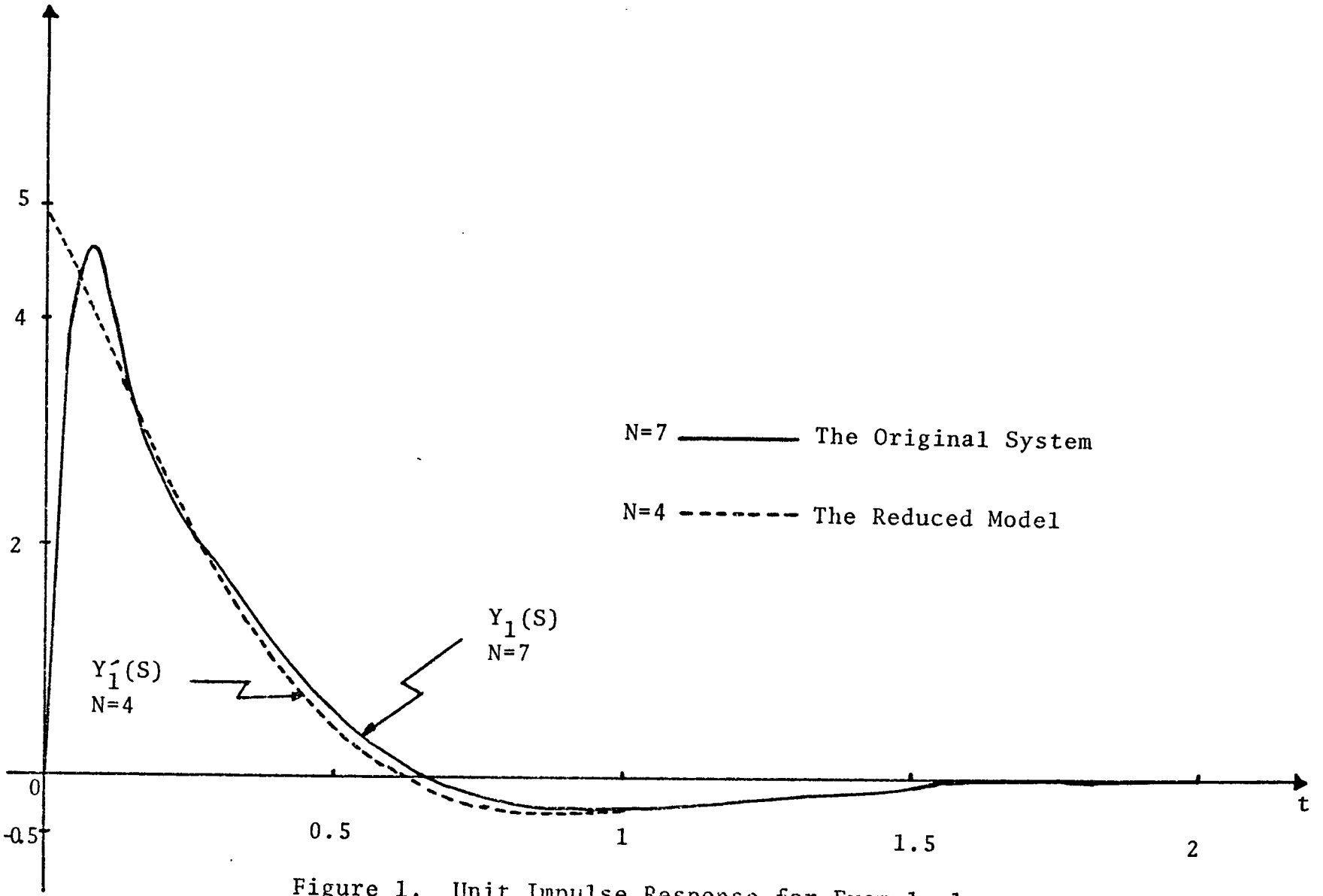

Figure 1. Unit Impulse Response for Example 1.

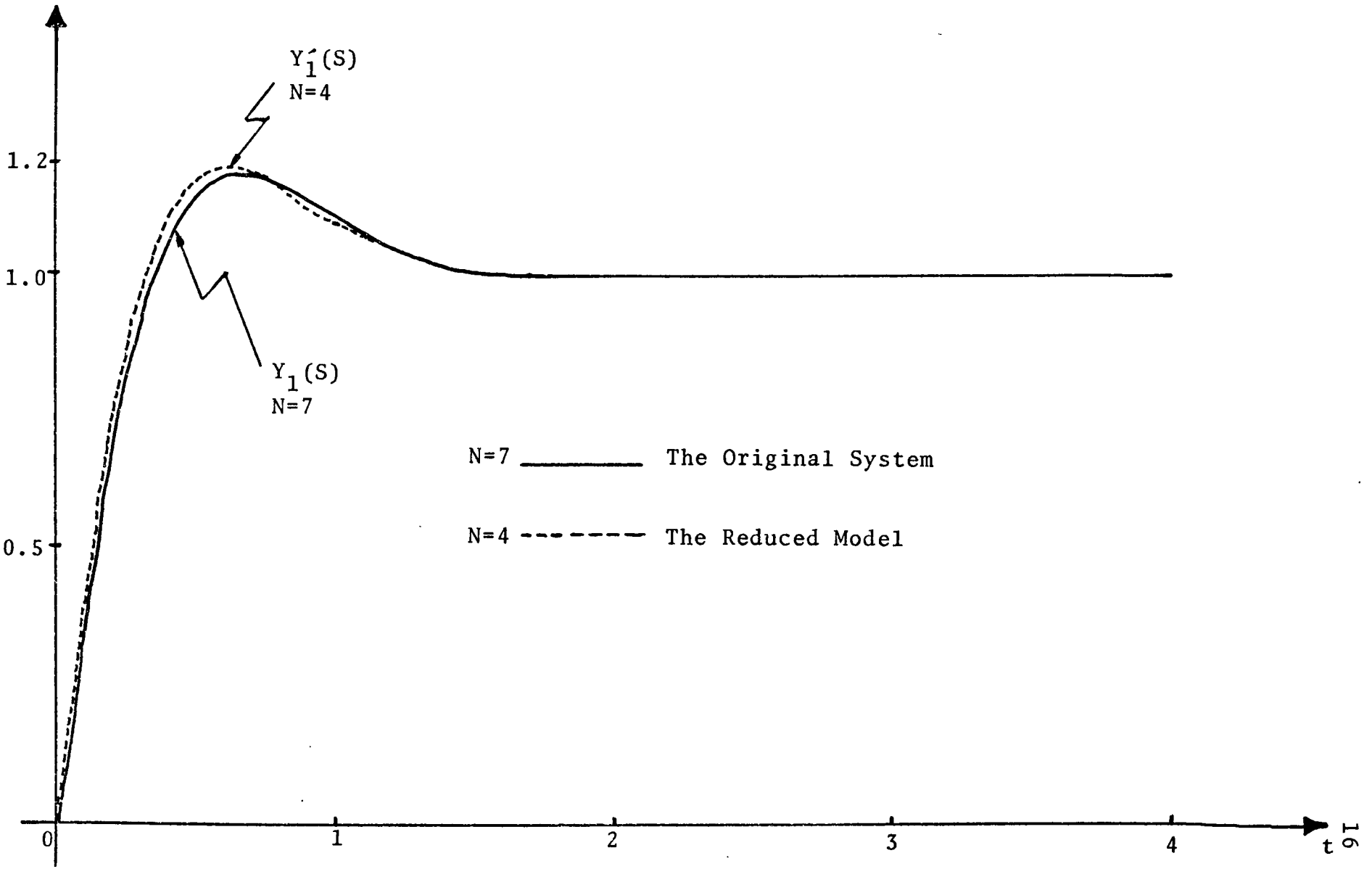

Figure 2. Unit Step Response for Example 1.

for which a third-order simplified model is desired.

Rewriting the numerator and denominator polynomials of Eq. (30) in ascending order and expanding according to Eq. (10), we again have 14 quotients, with the first three being:

$$h_1 = 1.0$$

$$h_2 = 4.119519$$

$$h_3 = -0.0660683$$

Factorize the characteristic equation, $\Delta(S)$, of Eq. (30) as:

$$\begin{aligned}\Delta(S) = {} & (S+1\cdot 89715385)(S+0\cdot 27276689+j1\cdot 04293823)\\ & (S+0\cdot 27276689-j1\cdot 04293823)(S+3\cdot 85106564+j9\cdot 65270519)\\ & (S+3\cdot 85106564-j9\cdot 65270519)(S+49\cdot 37869263)\\ & (S+52\cdot 51646423)\ .\end{aligned} \tag{31}$$

Then the characteristic equation of the desired reduced model is:

$$\begin{aligned}\Delta_4(S) &= (S+1\cdot 89715385)(S+0\cdot 27276689+j1\cdot 04293823)\\ &\quad (S+0\cdot 27276689-j1\cdot 04293823)\\ &= S^3+2\cdot 442688S^2+2\cdot 1970838S+2\cdot 204724\ .\end{aligned} \tag{32}$$

Applying again either Eq. (17) or Eq. (22), one can obtain the third-order reduced model

$$T(S) = \frac{C(S)}{R(S)} = \frac{0\cdot 072886S^2+1\cdot 6618942S+2\cdot 204724}{S^3+2\cdot 442688S^2+2\cdot 1970838S+2\cdot 204724} \tag{33}$$

The impulse responses of the original and approximated systems are shown in Fig. 3, and the unit step responses are shown in Fig. 4. It is noted that the simplification by the mixed method gives very close approximations.

For finding the integral square value (I.S.V.) of a response function T(S), Katz's method[11] can be applied. Following Katz's formula, we calculated the integral square values of examples 1 and 2 and obtained the following results:

Example 1	I.S.V.
Original model	3•413495
Fourth-order model	3•739678
Example 2	
Original model	1•269873
Third-order model	1•239319

(34)

From Eq. (34), we see that the mixed method reductions are satisfactory.

System 2. Multivariable System

In Section 1, the model-reduction techniques have been investigated for the single-input single-output systems. However, the single-variable system is a special case of a multivariable system. In general, control systems and other practical systems are high dimensional and with multi-input, multi-output. In this section, we shall concentrate on the reduction of multivariable systems. The model-reduction techniques developed in the previous section can be extended

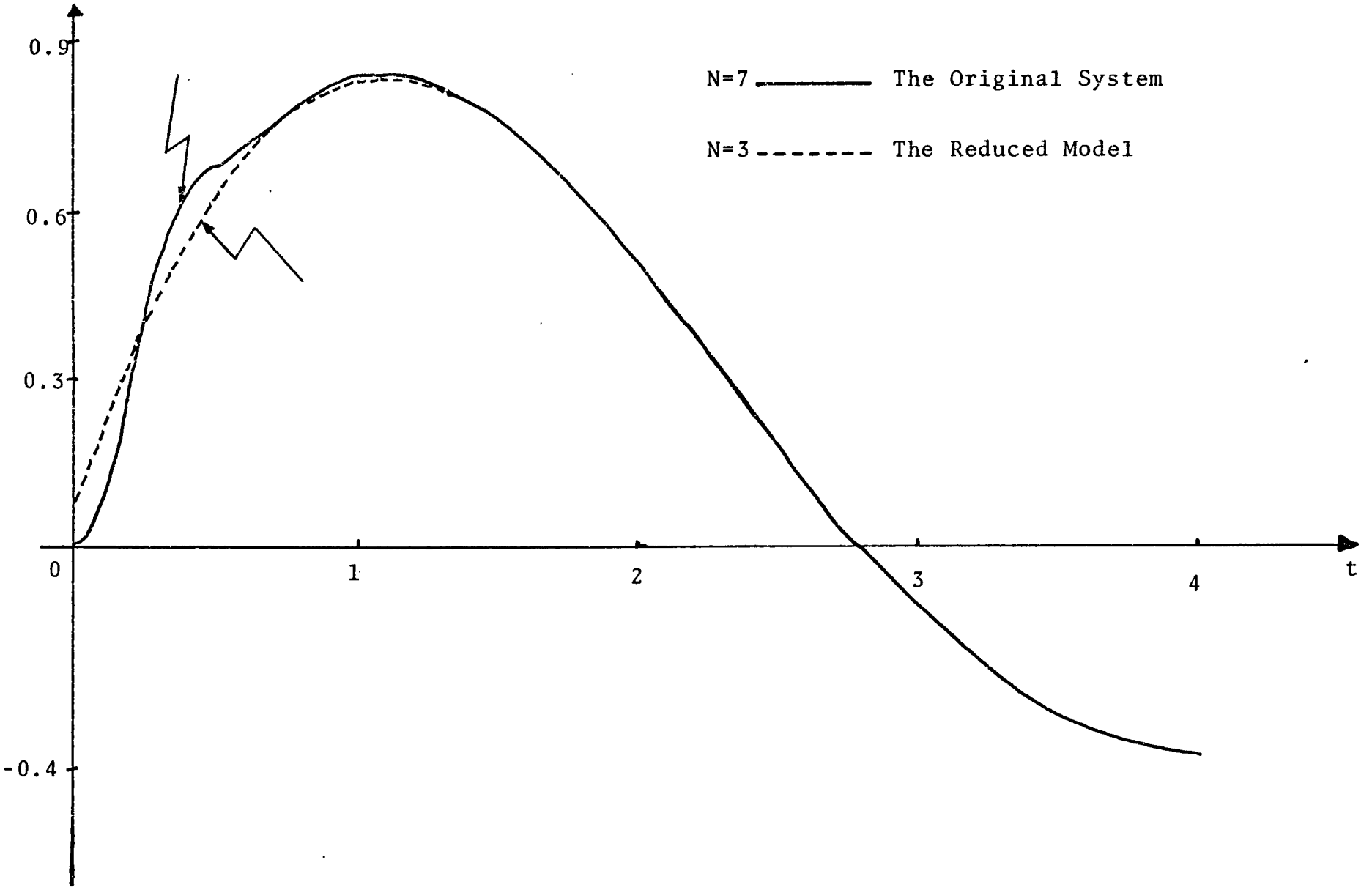

Figure 3. Unit Impulse Response for Example 2.

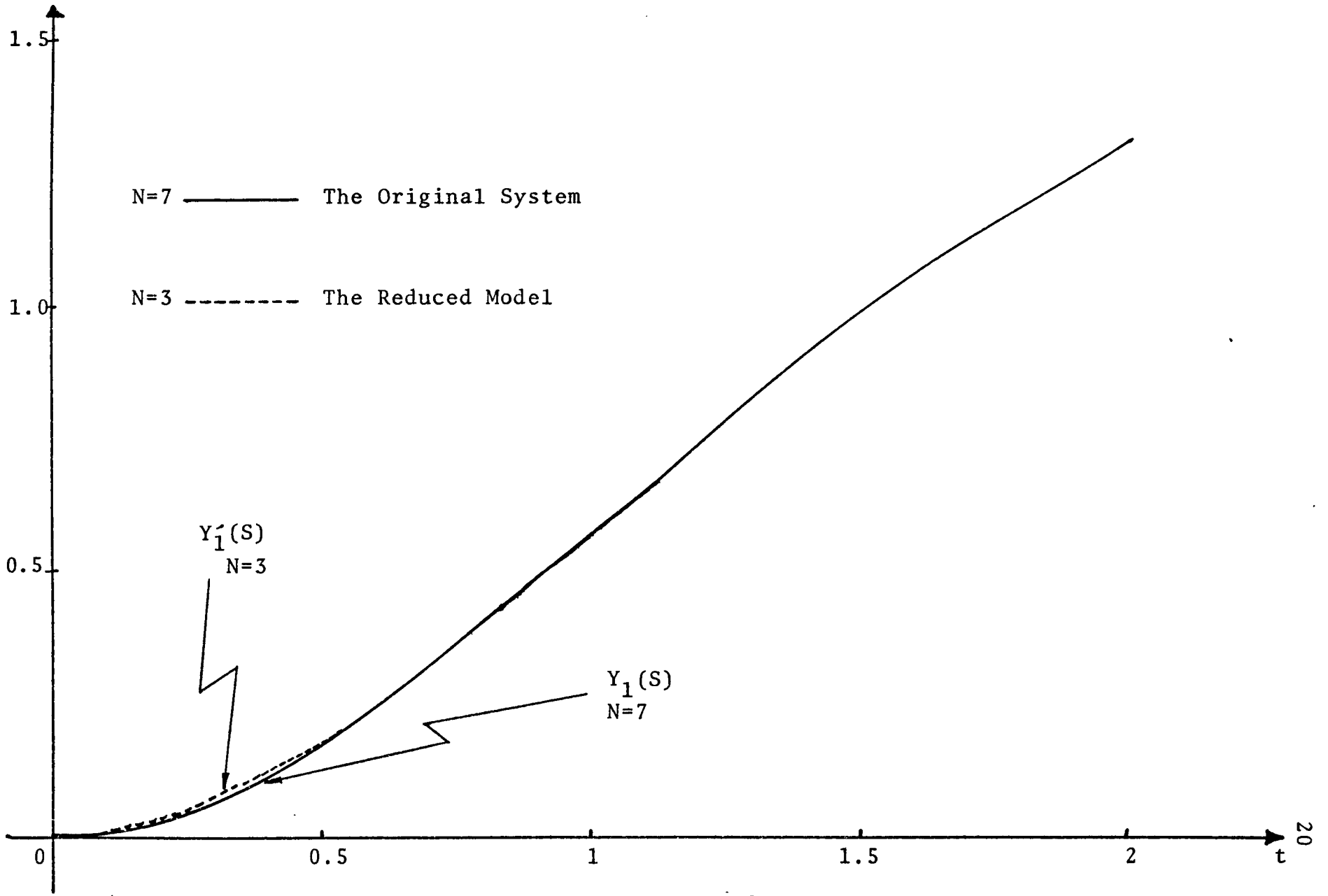

Figure 4. Unit Step Response for Example 2.

to multivariable system after some justification, such as replacing scaler variables by vector variables.

a) The Second Cauer Form

Consider the following transfer-function matrix of a multivariable system, T(S):

$$[T(S)] = \left[A_{21} + A_{22}S + A_{23}S^2 + \ldots + A_{2,n}S^{n-1}\right] \left[A'_{11} + A_{12}S + A_{13}S^2 + \ldots + A_{1,n+1}S^n\right]^{-1} \quad (35)$$

Where each $A_{i,j}$ is a real and constant m by m matrix. $A_{ij} = A_j[1]$, $j = 1,2,\ldots,n+1$ where A_j is a coefficient of the characteristic polynomial or $\Delta(S) = \sum_{j=1}^{n+1} A_j S^{j-1}$ and [1] is an identity matrix.

When the numbers of inputs equal that of the outputs, we can obtain the matrix quotients of Eq. (35) by means of the following matrix Routh algorithm:

$$\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & A_{14} & \ldots \\ A_{21} & A_{22} & A_{23} & \ldots & \\ A_{31} & A_{32} & \ldots & & \\ A_{41} & \ldots & & & \end{array} \quad (36)$$

The elements of the first and second rows of Eq. (36) are the matrix coefficients of the matrix transfer function of Eq. (35), and the elements of the third, fourth and subsequent rows can be evaluated by the following matrix Routh algorithm.[5,6]

$$A_{i,j} = A_{i-2,j+1} - H_{i-2}\, A_{i-1,j+1}; \quad i=3,4,\ldots \quad j=1,2,\ldots$$

$$H_i = A_{i,1}\left(A_{i+1,1}\right)^{-1} \quad ; \; i=1,2,\ldots, 2k \text{ and } k \leq n$$

$$\det\left(A_{i+1,1}\right) \neq 0 \tag{37}$$

The complete matrix Routh array is:

$$\begin{array}{lllllll}
 & & A_{11} & A_{12} & \cdots & A_{1,n} & A_{1,n+1} \\
H_1 = A_{11}A_{21}^{-1} & < & A_{21} & A_{22} & \cdots & A_{2,n} & \\
H_2 = A_{21}A_{31}^{-1} & < & A_{31} & A_{32} & \cdots & A_{3,n} & \\
H_3 = A_{31}A_{41}^{-1} & < & A_{41} & A_{42} & \cdots & & \\
H_4 = A_{41}A_{51}^{-1} & < & A_{51} & A_{52} & \cdots & & \\
 & & \cdots & & & &
\end{array} \tag{38}$$

Assumptions have been made that the first several dominant matrix quotients H_i exist, or $\det\,[A_{i+1,1}] \neq 0$, and a stable reduction can be obtained. These restrictions limit the applications of the methods of the matrix-continued fraction and the mixed method. The method proposed in Chapter IV will serve to eliminate these restrictions.

After the matrix quotients are found by applying Eq. (38), the second Caver matrix of Eq. (35) can be expanded according to Eq. (9) by replacing (h_i) with $[H_i]$ and using the matrix inversion rather than division. The resulting expansion of the matrix-continued fraction of the second Cauer matrix form is:

$$[T(S)] = \left[H_1 + \left[H_2\frac{1}{S} + \left[H_3 + \left[H_4\frac{1}{S} + [\ldots]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \tag{39}$$

where each $[H_i]$ is a matrix quotient of real and constant, m by m matrix. The block diagram corresponding to Eq. (39) is shown in Fig. 5.

The reduced model can be obtained by discarding the low performance matrix quotients located in the inner position of Eq. (39). The reduced models are

$$\begin{aligned} T_d(S) &\simeq \left[H_1 + \left[H_2\frac{1}{S} + \left[H_3 + \left[H_4\frac{1}{S}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \\ &= \left[\left(H_2 + H_4\right) S + H_2H_3H_4\right]\left[S^2I + \left(H_1H_2 + H_1H_4 \right.\right. \\ &\quad \left.\left. + H_3+H_4\right) S + H_1H_2H_3H_4\right]^{-1} \\ &\cong \left[H_1 + \left[H_2\frac{1}{S}\right]^{-1}\right]^{-1} \\ &= \left[H_2\right]\left[SI + H_1H_2\right]^{-1} \end{aligned} \tag{39a}$$

This reduced model gives a satisfactory approximation in the steady-state response.

If the matrix quotients, $H_{i,i=1,2,\ldots,2k}$, are given, Eq. (39) can be expressed by the following state equations.

$$\begin{aligned} [X] &= [A]\ [X] + [B]\ [U] \\ [Y] &= [C]^T\ [X] \end{aligned} \tag{40}$$

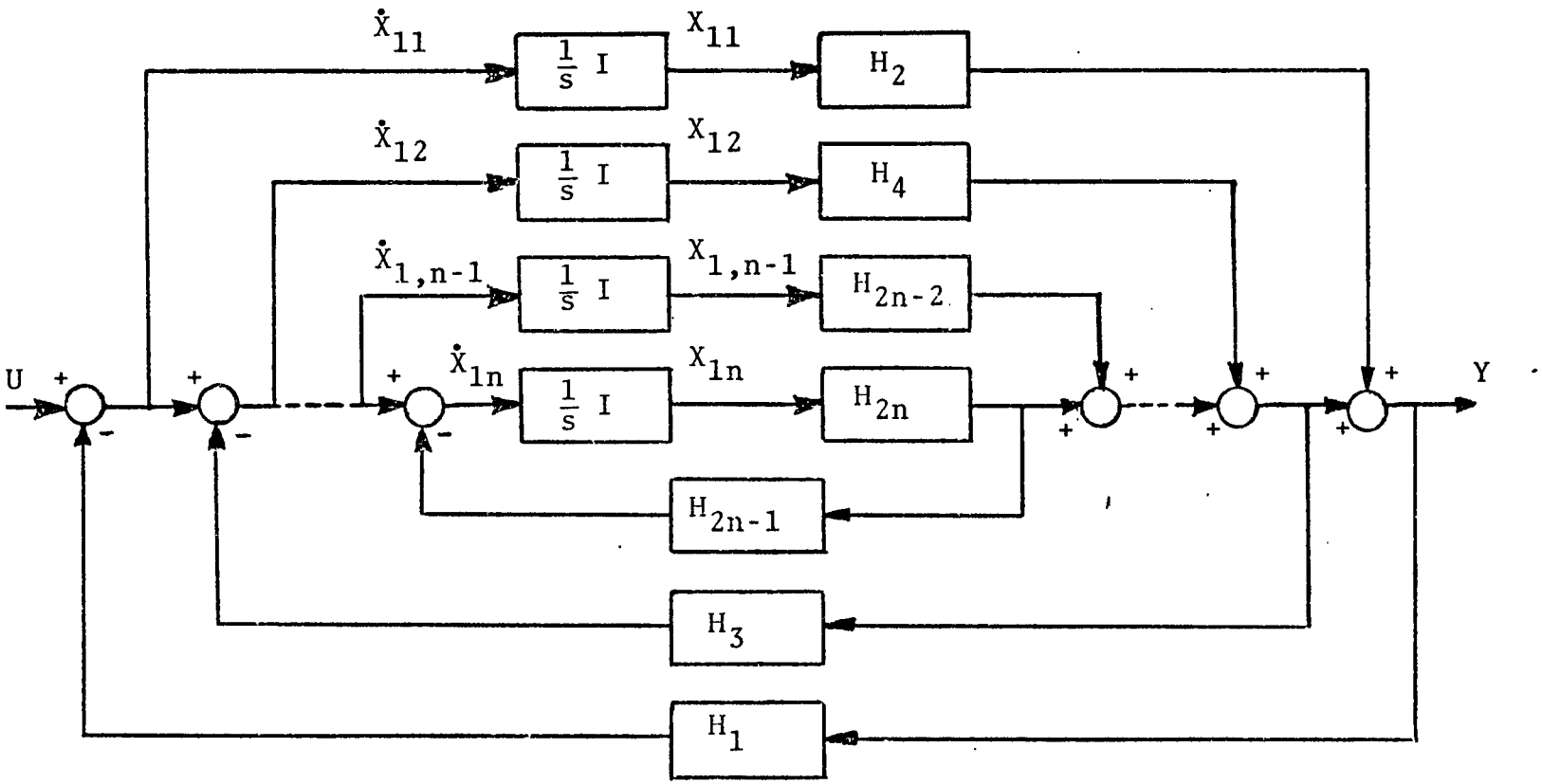

Figure 5. Block Diagram Representation of Second Cauer Matrix Form.

where

$$\overset{rxr}{[A]} = \begin{bmatrix} H_1H_2 & H_1H_4 & H_1H_6 & \cdots & H_1H_{2K} \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3)H_6 & \cdots & (H_1+H_3)H_{2K} \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3+H_5)H_6 & \cdots & (H_1+H_3+H_5)H_{2K} \\ \vdots & \vdots & \vdots & & \\ H_1H_2 & (H_1+H_3)H_4 & (H_1+H_3+H_5)H_6 & \cdots & (H_1+H_3+\cdots+H_{2K-1})H_{2K} \end{bmatrix} \tag{40a}$$

$$\overset{rxm}{[B]} = [I,I,\ldots,I]^T \tag{40b}$$

$$[C]^T = \left[H_2,H_4,\ldots H_{2K}\right] \tag{40c}$$

$$\overset{rx1}{[X]} = \left[X_{11},X_{12},\ldots,X_{1K}\right]^T$$

$$X_{11} = \left[X_1,X_2,\ldots X_m\right]^T$$

$$X_{12} = \left[X_{m+1},X_{m+2},\ldots X_{2m}\right]^T$$

$$\cdots$$

$$X_{1K} = \left[X_{(k-1)m+1},X_{(K-1)m+2},\ldots,X_{Km}\right]^T \tag{40d}$$

The matrix $\overset{rxr}{A}$ is a matrix with dimension rxr where γ=kxm, and K is an proper integer. The characteristic equation is $\Delta(S) = \prod_{i=1}^{r} (S-\lambda_i)$, where λ_i is the poles, [I] is an m by m dimensional-identity matrix, $[C]^T$ is an m by (kxm) matrix, [X] is an (Kxm]-dimensional state vector,

[U] and [Y] are m-dimensional input and output vector, respectively.

After substituting the matrix quotients into Eq. (40), the corresponding matrix transfer function can be obtained by applying the Leverrier algorithm.[12] The transfer function matrix has the form

$$T(S) = C^T (SI - A)^{-1} B = \frac{1}{\Delta(S)} [\Phi(S)] \tag{41}$$

where

$$\Delta(S) = S^r + d_1 S^{r-1} + d_2 S^{r-2} + \ldots + d_{r-1} S + d_r$$

$$[\Phi(S)] = S^{r-1} C^T B + S^{r-2} C^T R_1 B + \ldots + S C^T R_{r-2} B + C^T R_{r-1} B$$

and $d_1 = -tr(A)$ $\qquad R_1 = R_0 A + d_1 I$,

$d_2 = -\frac{1}{2} tr(R_1 A)$ $\qquad R_2 = R_1 A + d_2 I$

$\ldots$ $\qquad \ldots$

$d_{r-1} = -\frac{1}{r-1} tr(R_{r-2} A)$ $\qquad R_{r-1} = R_{r-2} A + d_{r-1} I$

$dr = -\frac{1}{r} tr(R_{r-1} A)$ $\qquad R_r = R_{r-1} A + d_r I = 0$

$t_r(A)$ is the trace of the matrix A; A, B, C, and R are constant, real matrices of compatible dimensions.

An alternate approach to obtain the matrix-continued fraction inversion or the corresponding matrix-transfer function can be evaluated from the following matrix Routh algorithm[10],

$$A_{2r+1,1} = [I]$$

$$A_{i,1} = H_i A_{i+1,1}; \quad i = 2r, 2r-1, \ldots, \ldots, 2, 1$$

$$A_{j-2,\ell+1} = A_{j,\ell} + H_{j-2} A_{j-1,\ell+1}; \quad j = 2r+1, 2r, \ldots, 3; \quad \ell = 1, 2, \ldots r \tag{42}$$

The matrix coefficients of Eq. (35) A_{1i}, A_{2i} are the elements of the first and second rows of the matrix Routh array generated by Eq. (42).

b) Mixed Method

As shown in Section 1, the reduced model in Eq. (39a) may be unstable, even if the original system is stable. The mixed method is presented for the reduction of multivariable systems which guarantees that the reduced model is stable and the dominant performance of the original system is maintained.

Let the multivariable system with m inputs and ℓ outputs be described by the matrix equation,

$$Y_0(S) = [T(S)] \; U_0(S) \tag{43}$$

The transfer-function matrix is,

$$[T(S)] = \frac{1}{\Delta_0(S)} [Q(S)] \tag{44}$$

where

$$\Delta_0(S) = \sum_{i=1}^{n+1} a_i S^{i-1} = \prod_{i=1}^{n} \left(S - \lambda_i\right); \quad a_1 \neq 0,$$

$$a_{n+1} = 1,$$

$$\lambda_i \neq 0 \text{ and}$$

$$[Q(S)] = \sum_{i=1}^{n} Q_i S^{i-1}$$

In this section, we consider the case $\ell = m$. The steps involved in the mixed method for the multivariable system reduction can be summarized as follows.

<u>Step 1</u>. Determine the characteristic polynomial of the matrix [T(S)] by applying Gilbert's method.[13] The characteristic polynomial of the matrix [T(S)] is

$$\Delta(S) = \left(S - \lambda_1\right)^{\gamma_1} \left(S - \lambda_2\right)^{\gamma_2} \ldots \left(S - \lambda_n\right)^{\gamma_n} \tag{45}$$

If p dominant poles which have $\gamma_i = m$ repeated power are chosen as the dominant eigenvalues of the reduced model, then the least common-denominator polynomial $\Delta p(S)$ and the characteristic polynomial $\Delta^c p(S)$ are written:

$$\Delta p(S) = \prod_{i=1}^{p} \left(S - \lambda_i\right) = \sum_{j=1}^{p+1} d_j S^{j-1}, \quad d_{p+1=1} \tag{46}$$

$$\Delta_p^c(S) = \prod_{i=1}^{p} \left(S - \lambda_i\right)^m \tag{47}$$

Step 2. Use the dominant matrix quotients $H_{i,i=1,2,\ldots,}$ obtained by Eq. (37) and apply the following algorithm to fit the numerator dynamics

$(B_{2,j},\ j=1,2,\ldots,p)$ of the reduced model.

$$B_{1,j} = d_j[I],\ j=1,2,\ldots,p$$

$$B_{1,p+1} = [I]$$

$$B_{i+1,1} = H_i^{-1}\ B_{i,1} \quad ; \quad i=1,2,\ldots,p \text{ and } p \leq n$$

$$B_{i+1,j+1} = H_i^{-1}\left[B_{i,j+1} - B_{i+2,j}\right] : \quad i=1,2,\ldots,p-j; \quad j=1,2,\ldots,p-1 \tag{48}$$

The reduced model is:

$$\begin{aligned} Y_d(S) &= \left[T_d(S)\right]\ \left[U_0(S)\right] \\ &\simeq [Y_0(S)] = [T(S)]\ \left[U_0(S)\right] \end{aligned} \tag{49}$$

where

$$T_d(S) = \frac{1}{\Delta_p(S)}\left[\sum_{j=1}^{p} B_{2,j}S^{j-1}\right]$$

Again there is an alternative Routh algorithm for the multivariable system which yields identical results. Replacing h_i, b_{1j} and b_{2j} by H_i, B_{1j} and B_{2j}, respectively, in Eq. (18) yields

$$
\begin{matrix}
B_{11} & B_{12} & B_{13} & \cdots \\
B_{21} & B_{22} & B_{23} & \cdots \\
B_{12}-H_1B_{22} & B_{13}-H_1B_{23} & B_{14}-H_1\ \ 24 & \cdots \\
B_{22}-H_2\left(B_{13}-H_1B_{23}\right) & 23-H_2\left(B_{14}-H_1B_{24}\right) & \cdots & \cdots
\end{matrix}
\tag{50}
$$

It is observed that

$$
\begin{aligned}
H_1 &= B_{11}B_{21}{}^{-1} \\
H_2 &= B_{21}\ B_{12}-H_1B_{22}\ {}^{-1} \\
H_3 &= \left[B_{12}-H_1B_{22}\right]\left[B_{22}-H_2\left(B_{13}-H_1B_{23}\right)\right]^{-1} \\
H_4 &= \left[B_{22}-H_2\left(B_{13}-H_1B_{23}\right)\right]\left[B_{13}-H_1B_{23}-H_3\left[B_{23}-H_2\left(B_{14}-H_1B_{24}\right)\right]\right]^{-1}
\end{aligned}
\tag{51}
$$

After some manipulation of Eq. (51), the following matrix expression can be formulated:

$$
\begin{bmatrix}
I & 0 & 0 & 0 & \cdots \\
0 & H_2 & 0 & 0 & \cdots \\
0 & I & H_3H_2 & 0 & \cdots \\
0 & 0 & H_2+H_4 & H_4H_3H_2 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{bmatrix}
\begin{bmatrix}
B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \\ \cdot
\end{bmatrix}
$$

$$= \begin{bmatrix} H_1 & 0 & 0 & 0 & \cdots \\ I & H_2H_1 & 0 & 0 & \cdots \\ 0 & H_1+H_3 & H_3H_2H_1 & 0 & \cdots \\ 0 & I & H_2H_1+H_4H_1+H_4H_3 & H_4H_3H_2H_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \\ B_{23} \\ B_{24} \\ \cdot \end{bmatrix} \tag{52}$$

The factored form of Eq. (52) is shown in Eq. (53)

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & H_2 & 0 & \cdots & 0 & 0 \\ 0 & I & H_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & H_p \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & H_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & H_{p-1} \end{bmatrix} \cdots \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & H_2 \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{12} \\ B_{13} \\ \vdots \\ B_{1,p} \end{bmatrix}$$

$$= \begin{bmatrix} H_1 & 0 & \cdots & 0 & 0 \\ I & H_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & H_p \end{bmatrix} \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & H_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & H_{p-1} \end{bmatrix} \cdots \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & H_1 \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2,p} \end{bmatrix} \tag{53}$$

where p is the number of dominant poles chosen for the desired reduced model and $B_{1,p} = [I]$. [I] is an identity matrix with compatible dimension.

Eq. (53) can be written as:

$$H_L B_1 = H_R B_2 \tag{54}$$

where H_L and H_R are the product of bidiagonal matrices. If $H_{i\neq 0}, i=1,2,\ldots, \rho$, both H_L and H_R are nonsingular. Eq. (54) can be further simplified as:

$$B_2 = H\ B_1 \tag{55}$$

where $H = H_R^{-1}\ H_L$. By using Eq. (55), $B_{2,j}$ can be computed if $B_{i,j}$ and H_j are given.

It should be noted[6] that Eq. (40) is a minimal realization of T(S) and the minimal dimension of the system matrix is γ = KXM, where K is the smaller integer and m is the dimension of the matrix quotients. This observation can be easily verified by Gilbert's theorem.[13] For example, starting from the innermost loop of the block diagram shown in Fig. 5, we find that the subsystem in the forward path $\frac{1}{S}\ [H_{2K}]$ has minimal dimension m if and only if det $[H_{2K}] \neq 0$. The subsystem in the feedback path $[H_{2K-1}]$ and the forward subsystem $\frac{1}{S}\ [H_{2K}]$ form a composite feedback system. It is completely controllable and observable if det $[H_{2K}] \neq 0$ and det $[H_{2K-1}] \neq 0$ because det $[H_{2K-1}\ H_{2K}]$ = det $[H_{2K}\ H_{2K-1}]$ = det $[H_{2K}]$ det $[H_{2K-1}]$. The minimal dimension of this feedback system is m. Furthermore, this composite feedback subsystem and the other feedforward path $\frac{1}{S}\ [H_{2K-2}]$ form a parallel connection. If det $[H_{2K-2}] \neq 0$, then the parallel

subsystems, which have no common pole at $S = 0$, form an irreducible subsystem with minimal dimension 2M. By extending this approach to the whole system, we can conclude that the system of Eq. (40) is completely controllable and observable with minimal dimension $\gamma = KXM$ if $\det [H_i] \neq 0$, $i=1,2,\ldots,2K$ and $K \leq n$.

From this conclusion, we obtain a sufficient condition that when 2K matrix quotients are available, the minimal dimension of the realization is KXM. On the other hand, if the rank of a transfer-function matrix equals to KXM where K is the smaller integer, then a complete set of matrix quotients can be obtained.

In short, for a system with rank = γ, dimension of $H_i = m$ then $\frac{\gamma}{m} = K \nRightarrow 2K$ matrix quotients are expected.

Before proceeding, we make the following definition[12]:

> The characteristic polynomial of a proper rational matrix T(S) is defined to be the least common denominator of all minors of T(S). The degree of the characteristic polynomial of T(S) is equal to the rank of T(S).

It is noted that the common denominator polynomial is not necessarily the characteristic polynomial. The absolute stability of a multivariable system can be determined by applying the Routh criterion[14] to the characteristic equation obtained.

A method for computing the correct rank of a given matrix-transfer function for the purpose of constructing the state variable representation is contained in an important theorem by Gilbert[13].

Theorem: Given a rational, proper matrix transfer function, T(S), whose elements have a finite number of simple poles, S_i, i=1,2,...,n. The partial fraction expansion of T(S) can be expressed as

$$T(S) = \sum_{i=1}^{n} \frac{D_i}{S - S_i} + R \tag{56}$$

where

$$D_i = \lim_{S \to S_i} \left[\left(S - S_i \right) T(S) \right]$$

$$R = \lim_{S \to \infty} T(S)$$

Let the rank of matrix D_i be denoted by γ_i, then T(S) can be represented by a system of differential equation, Eq. (40), whose rank is

$$\gamma = \sum_{i=1}^{n} \gamma_i \tag{57}$$

The following examples illustrate the power of the mixed method proposed in this research.

Example 3

Consider the transfer-function matrix

$$T(S) = \frac{\begin{bmatrix} 15\cdot0(S+1\cdot7)(S+100) & 95200\cdot0(S+1\cdot898)(S+10) \\ 85\cdot0(S+1\cdot44)(S+100) & 124000\cdot0(S+2\cdot037)(S+10) \end{bmatrix}}{(S+1\cdot338354)(S+1\cdot886647)(S+10\cdot0)(S+100\cdot0)} \tag{58}$$

The above matrix equation is written in the form of a partial fraction expansion to check the rank.

$$T(S) = \sum_{i=1}^{4} \frac{D_i}{(S - S_i)} \tag{59}$$

where D_i is the ith residue matrix given by Eq. (56) and S_i, the poles of the matrix elements in T(S), are

$$S_1 = -1\cdot338354 \qquad S_2 = -1\cdot886647$$

$$S_3 = -10\cdot0 \qquad S_4 = -100\cdot0 \tag{60a}$$

and

$$D_1 = \begin{bmatrix} 535\cdot20886 & 461477\cdot71 \\ 852\cdot42777 & 750376\cdot53 \end{bmatrix} \equiv \text{rank two}$$

$$D_2 = \begin{bmatrix} -274\cdot68846 & 8768\cdot9573 \\ -3724\cdot8732 & 151263\cdot5 \end{bmatrix} \equiv \text{rank two}$$

$$D_3 = \begin{bmatrix} -11205\cdot0 & 0 \\ -65484\cdot0 & 0 \end{bmatrix} \equiv \text{rank one}$$

$$D_4 = \begin{bmatrix} 0 & 840537930\cdot0 \\ 0 & 1093267000\cdot0 \end{bmatrix} \equiv \text{rank one} \tag{60b}$$

The system is obviously of rank six, with

$$S_1 = \lambda_1 = -\ 1{\cdot}338354 \equiv \text{pole of order two}$$

$$S_2 = \lambda_2 = -1{\cdot}886657 \equiv \text{pole of order two}$$

$$S_3 = \lambda_3 = -10{\cdot}0 \equiv \text{pole of order one}$$

$$S_4 = \lambda_4 = =100{\cdot}0 \equiv \text{pole of order one} \qquad (61)$$

It is noted that $K = \frac{\gamma}{m}$ where m=2 for this particular system. Hence $K = \frac{6}{2} = 3$ and $2 \times 3 = 6$ matrix quotients are expected. Eventually, a yield of six matrix quotients are obtained by applying Eq. (37).

$$H_1 = \begin{bmatrix} -0{\cdot}40686947 & 0{\cdot}29105532 \\ 0{\cdot}0019716227 & -0{\cdot}41075473 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1{\cdot}2684087 & 941{\cdot}97892 \\ 7{\cdot}1738115 & 1240{\cdot}9788 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 3{\cdot}124612 & -2{\cdot}1961065 \\ -0{\cdot}14323896 & 0{\cdot}025205154 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 2{\cdot}4259475 & 6048{\cdot}3982 \\ -1{\cdot}4945432 & 8901{\cdot}132 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} 5{\cdot}9614921 & -4{\cdot}1304099 \\ -0{\cdot}00286802 & 0{\cdot}00201857 \end{bmatrix}, \quad H_6 = \begin{bmatrix} -3{\cdot}6943562 & -6990{\cdot}3771 \\ -5{\cdot}6792683 & -10142{\cdot}111 \end{bmatrix} \qquad (62)$$

The reduced model of this system is found by computing the approximated denominator polynomial

$$A_2(S) = (S+1{\cdot}338354)\ (S+1{\cdot}886647)$$

$$= S^2+3{\cdot}225S + 2{\cdot}525 \qquad (63)$$

Only the first two matrix quotients are used to evaluate the approximate numerator. Substituting the $B_{1,j}$ and H_i into Eq. (48) or Eq. (55) yields $B_{2,j}$. The reduced model is

$$\left[T_2(S)\right] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 935\cdot17604S + 1809\cdot446 \\ 1222\cdot0172S + 2538\cdot12 \end{bmatrix}$$

The unit-step response curves of the original system and the reduced model are compared in Fig. 6. The I.S.V. of the original system and the second-order approximated model are:

	I.S.V.
First curve of original model	337408·375
Second curve of original model	628904·5
First curve of reduced model	336624·0625
Second curve of reduced model	627075·0625

(65)

demonstrating that the approximation is very satisfactory.

Example 4

Slightly modify the matrix transfer function in example 3 as:

$$T(S) = \frac{\begin{bmatrix} 15\cdot0(S+1\cdot7)(S+100\cdot1) & 95200\cdot0(S+1\cdot898)(S+10\cdot0) \\ 85\cdot0(S+1\cdot44)(S+100) & 124000\cdot0(S+2\cdot037)(S+10\cdot1) \end{bmatrix}}{(S+1\cdot338354)(S+1\cdot886647)(S+10\cdot0)(S+100\cdot0)} \tag{66}$$

Applying the same procedures as in example 3 shows that the rank of the system is now eight instead of six, meaning that each pole is of order two. It should note again that

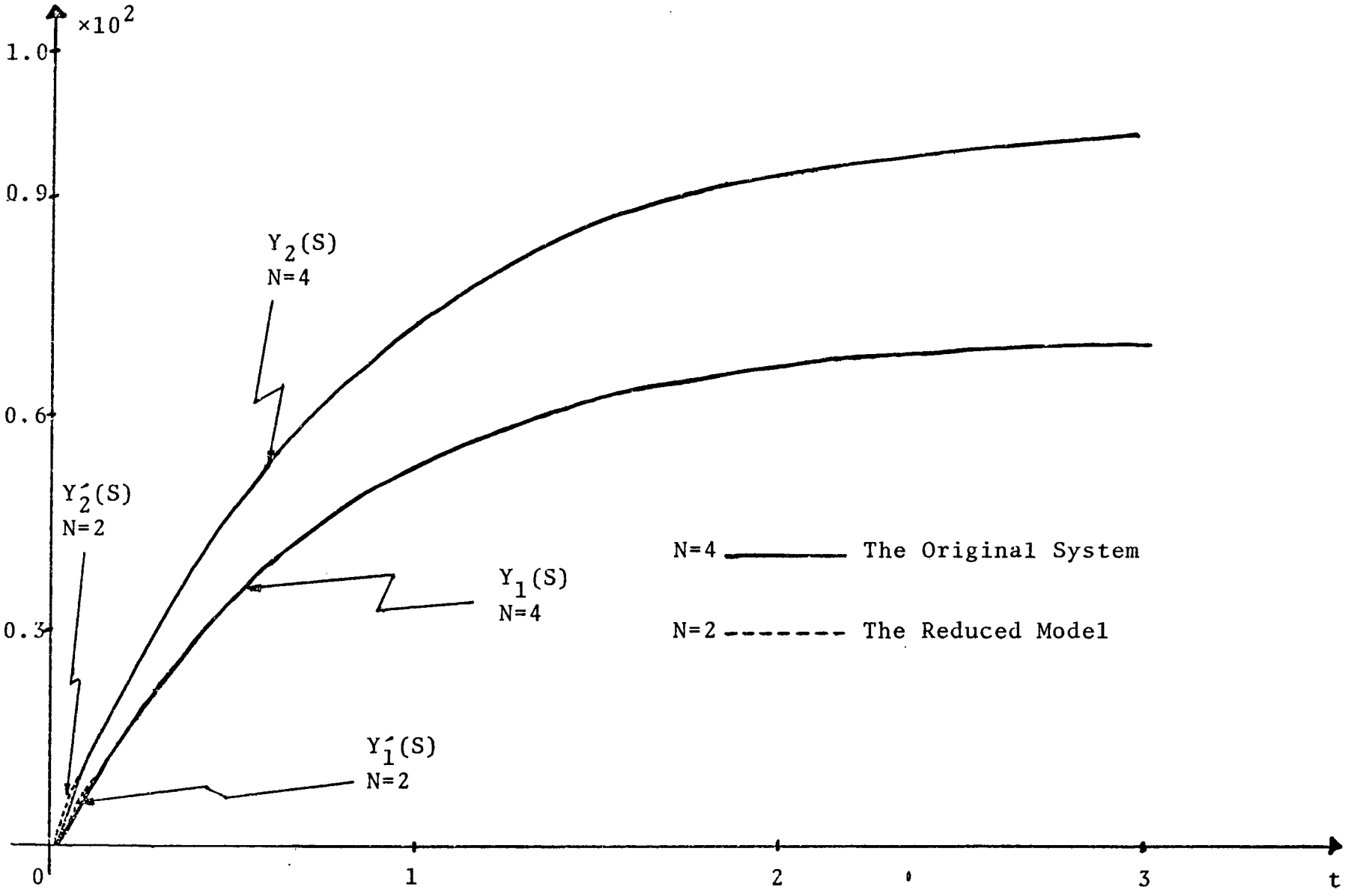

Figure 6. Unit Step Response for Example 3.

$K = \frac{8}{2} = 4 = 2 \times 4 = 8$ matrix quotients. Rearranging Eq. (66) in ascending order of the form Eq. (44)

$$T(S) = \frac{\begin{bmatrix} 2552\cdot55 & 1806896\cdot0 \\ 12240\cdot0 & 2551138\cdot8 \end{bmatrix} + \begin{bmatrix} 1527\cdot0 & 1132689\cdot6 \\ 8622\cdot4 & 1504988\cdot0 \end{bmatrix} S + \begin{bmatrix} 15\cdot0 & 95200\cdot0 \\ 85\cdot0 & 124000\cdot0 \end{bmatrix} S^2}{2525\cdot0+43502\cdot75S+1357\cdot275S^2+113\cdot225S^3+S^4} \tag{67}$$

The required H_i, $i=1,2,\ldots,8$ can be evaluated by applying Eq. (37) to Eq. (67). The eight matrix quotients are

$$H_1 = \begin{bmatrix} -0\cdot41280569 & 0\cdot29237804 \\ 0\cdot0019805828 & -0\cdot00041303 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1\cdot2695779 & 942\cdot01771 \\ 7\cdot1757979 & 1251\cdot1633 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 3\cdot3877287 & -2\cdot2360282 \\ -0\cdot14705349 & 0\cdot025832816 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1\cdot1284751 & 2561\cdot2092 \\ -3\cdot0552006 & 3993\cdot504 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} 3\cdot4149102 & -2\cdot3468879 \\ -0\cdot0013859 & 0\cdot00113316 \end{bmatrix}, \quad H_6 = \begin{bmatrix} -2\cdot3979783 & -3671\cdot6249 \\ -4\cdot1203335 & -5463\cdot4547 \end{bmatrix}$$

$$H_7 = \begin{bmatrix} 778453\cdot35 & -599165\cdot77 \\ 0\cdot24001049 & -0\cdot14767886 \end{bmatrix}, \quad H_8 = \begin{bmatrix} -0\cdot0000746 & 168\cdot398 \\ -0\cdot0002638 & 218\cdot78744 \end{bmatrix} \tag{68}$$

If a second order of reduction is desired, then the approximated denominator polynomial is

$$\begin{aligned} \Delta_2(S) &= (S+1\cdot338354)(S+1\cdot886647) \\ &= S^2+3\cdot225S+2\cdot525 \end{aligned} \tag{69}$$

Substituting $B_{1,1}$, $B_{1,2}$, $B_{1,3}$ and H_1, H_2 Into Eq. (48) or Eq. (55), the approximate numerator is

$$T_2(S) = \frac{\begin{bmatrix} 1\cdot2462196 & 933\cdot93105 \\ 7\cdot276 & 1224\cdot3628 \end{bmatrix} S + \begin{bmatrix} 2\cdot55255 & 1806\cdot896 \\ 12\cdot24 & 2551\cdot1388 \end{bmatrix}}{S^2+3\cdot225S+2\cdot525} \tag{70}$$

If we expand the original matrix equation into scaler input-output expressions, the unit step responses are:

$$T_1(S) = \frac{95215\cdot 0S^2+1134216\cdot 6S+1809448\cdot 55}{S^4+113\cdot 225S^3+1357\cdot 275S^2+3502\cdot 75S+2525\cdot 0} \cdot \frac{1}{S} \qquad (71a)$$

$$T_2(S) = \frac{124085\cdot 0S^2+1513610\cdot 4S+2563378\cdot 8}{S^4+113\cdot 225S^3+1357\cdot 275S^2+3502\cdot 75S+2525\cdot 0} \cdot \frac{1}{S} \qquad (71b)$$

and the unit step responses of the reduced model are:

$$T\alpha_1(S) = \frac{935\cdot 17726S+1809\cdot 4485}{S^2+3\cdot 225S+2\cdot 525} \cdot \frac{1}{S} \qquad (72a)$$

$$T\alpha_2(S) \quad \frac{1231\cdot 6388S+2563\cdot 3788}{S^2+3\cdot 225S+2\cdot 525} \cdot \frac{1}{S} \qquad (72b)$$

The comparison of T(t) and Tα(t) are shown graphically in Fig. 7. The steady-state responses are reproduced exactly, while the initial responses of the original model and the reduced model are also very close. Numerically, I.S.V. are:

	I.S.V.
First curve of original model	337409·25
Second curve of original model	640235·0
First curve of reduced model	336625·0
Second curve of reduced model	638647·25

(73)

From the above two examples, we see that the characteristic polynomial of T(S) is in general different from the common denominator polynomial of the determinant of T(S)

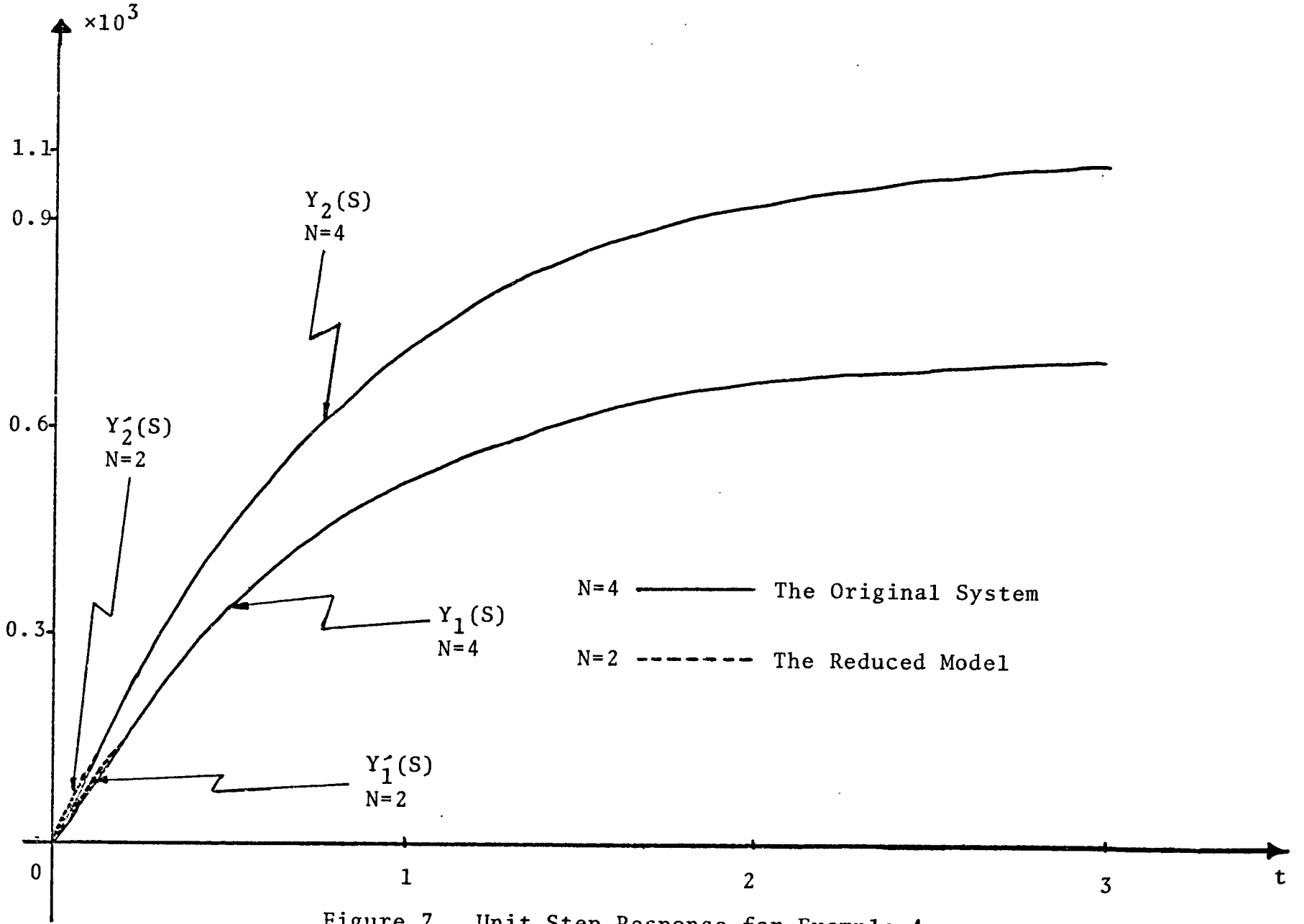

Figure 7. Unit Step Response for Example 4.

[if T(S) is a square matrix]. If T(S) is scaler (a 1 × 1 matrix), the denominator of T(S) and the characteristic polynomial of T(S) would be the same.

It should be pointed out that the rank of Eq. (44) must first be known; otherwise it is possible for the denominator of the reduced model to be of a higher order than the given system.

The following example will show that the accuracy of approximation by the mixed method depends upon the numbers of dominant-matrix quotients used.

Example 5

Consider that a reduced model of the following high-order transfer-function matrix is required:

$$[T(S)] = \frac{1}{\Delta(S)} \begin{bmatrix} T_{11}(S) & T_{12}(S) \\ T_{21}(S) & T_{22}(S) \end{bmatrix} \tag{74}$$

where $\Delta(S) = S^7+0\cdot258656\times10^3S^6+0\cdot43096293\times10^6S^5+0\times48281779\times10^8S^4+0\cdot18443232\times10^{10}S^3+0\cdot25036464\times10^{11}S^2+0\cdot5465822\times10^{11}S+0\cdot11861872\times10^{11}$

$T_{11}(S) = -0\cdot12413777\times10^2S^5+0\cdot12124793\times10^5S^4-0\cdot28814104\times10^7S^3-0\cdot33688681\times10^9S^2-0\cdot65066826\times10^{10}S-0\cdot34016025\times10^{11}$

$T_{12}(S) = 0\cdot5208\times10^2S^6+0\cdot10758478\times10^5S^5+0\cdot21869383\times10^8S^4+0\cdot13737148\times10^{10}S^3+0\cdot218624\times10^{11}S^2+0\cdot16562047\times10^{11}S+0\cdot25930241\times10^{11}$

$$T_{21}(S) = 0\cdot20045556S^6+0\cdot4786274\times10^2S^5+0\cdot39267902\times10^5S^4 +0\cdot51539505\times10^7S^3+0\cdot23138407\times10^9S^2+0\cdot34829993\times10^{10}S +0\cdot55728265\times10^{10}$$

$$T_{22}(S) = 0\cdot74534552\times10S^5+0\cdot27013426\times10^5S^4+0\cdot86810946\times10^6S^3 -0\cdot16222221\times10^8S^2-0\cdot63230209\times10^9S-0\cdot84962741\times10^{10}$$

The eigenvalues of the original system are

$$\lambda_1 = -0\cdot24374787 \ , \ \lambda_2 = -2\cdot3731406$$
$$\lambda_3 = -27\cdot890472 \ , \ \lambda_4 = -36\cdot71703$$
$$\lambda_5 = -48\cdot848372 \ , \ \lambda_6 = -71\cdot291619-j636\cdot28052$$
$$\lambda_7 = -71\cdot291619+j636\cdot28052 \tag{75}$$

By selecting various combination of dominant eigenvalues, we have the following approximated denominator polynomials.

(i) If $p = 3$ and λ_1, λ_2 and λ_3 are used as dominant eigenvalues, then the simplified denominator polynomial is

$$\Delta_3(S) = S^3+30\cdot50736047S^2+73\cdot56470257S+16\cdot13318683 \tag{76}$$

(ii) If $p = 4$ and λ_1, λ_2, λ_3 and λ_4 are used, we have

$$\Delta_4(S) = S^4+67\cdot22439047S^3+1193\cdot704373S^2+2717\cdot210578S +592\cdot3627048 \tag{76a}$$

The rank of [T(S)] is 15. The ratio of the rank and the dimension of this modified system is $\frac{14}{2} = 7 = K$, an integer. We have $2K = 14$ matrix quotients. The required

H_i, i=1,2,...,p can be evaluated by applying Eq. (37) to Eq. (74). The first four matrix quotients are

$$H_1 = \begin{bmatrix} -0\cdot69742845 & -2\cdot1285198 \\ -0\cdot4574532 & -2\cdot7922526 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0\cdot65449285 & 0\cdot53927311 \\ 0\cdot11563255 & -0\cdot16002077 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 6\cdot7166039 & -12\cdot832236 \\ -0\cdot6360288 & 20\cdot365442 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0\cdot44788433 & 1\cdot4644699 \\ 0\cdot027248267 & 0\cdot15292068 \end{bmatrix} \tag{77}$$

Substituting the $A_{1,j}$ and H_i obtained in Eqs. (76) and (77) into Eq. (48) yields $A_{2,j}$. The reduced models are

$$T_3(S) = \frac{1}{\Delta_4(S)} \begin{bmatrix} -0\ 11309S^2-6\ 62642S\ , & 28\ 6k34S^2+20\ 83107S \\ -46\ 26478 & +35\ 2674 \\ 0\ 10019S^2+4\ 37295S\ , & -0\ 75984\times10^{-3}S^2-0\ 30462S \\ +7\ 5795 & -11\ 5557 \end{bmatrix} \tag{78}$$

$$T_4(S) = \frac{1}{\Delta_4(S)} \begin{bmatrix} 0\cdot84\times10^{-1}S^3-0\cdot1078\times10^2S^2, & 0\cdot462\times10^2S^2+0\cdot1075\times10^4S^2 \\ -0\cdot28956\times10^3S-0\cdot16987\times10^4 & +0\cdot8001\times10^3S+0\cdot12949\times10^4 \\ 0\cdot88\times10^{-1}S^3+0\cdot8052\times10S^2, & 0\cdot51\times10^{-1}S^3-0\cdot333S^2 \\ +0\cdot168\times10^3S+0\cdot278\times10^3 & -0\cdot227\times10^2S-0\cdot424\times10^3 \end{bmatrix} \tag{79}$$

The unit-step response curve of the original system and the reduced models are compared in Fig. 8, and again the approximation is very satisfactory. Note that the higher the order of the reduced model, the better is the approximation.

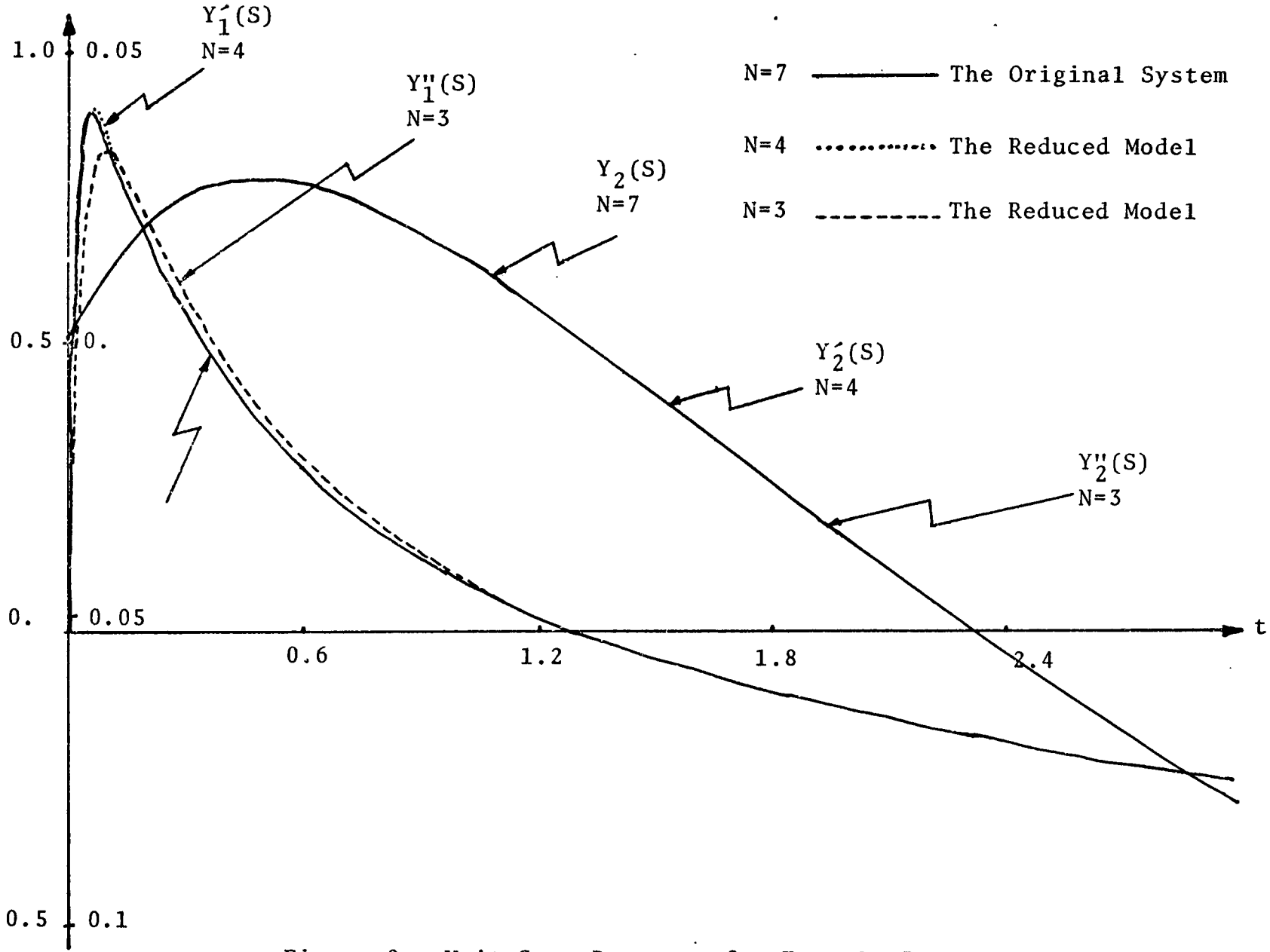

Figure 8. Unit Step Response for Example 5.

The I.S.V. of this problem are stated below:

	I.S.V.
First curve of original model	25·9255
Second curve of original model	$10\cdot7755\times10^{-2}$
First curve of third-order model	13·74108
Second curve of third-order model	$10\cdot7634\times10^{-2}$
First curve of fourth-order model	23·82907
Second curve of fourth-order model	$10\cdot7667\times10^{-2}$

Example 6

Consider another high-order multivariable system

$$[T(S)] = \frac{1}{\Delta(S)}\begin{bmatrix} T_{11}(S) & T_{12}(S) \\ T_{21}(S) & T_{22}(S) \end{bmatrix}$$

where $\Delta(S) = S^8+67\cdot8S^7+1285\cdot4S^6+8976\cdot1S^5+38697\cdot4S^4$
$+105846\cdot1S^3+159414\cdot8S^2+114239\cdot1S+30208\cdot2$
$= (S+0\cdot69302487)(S+0\cdot86648517)(S+1\cdot2965031$
$+j3\cdot49839115)(S+1\cdot296503-j3\cdot49839115)$
$(S+1\cdot81142425)(S+2\cdot73775863)(S+17\cdot53158569)$
$(S+41\cdot56671142)$

$T_{11}(S) = -4\cdot3S^7-260\cdot7S^6-4192\cdot2S^5-16306\cdot1S^4-50607\cdot4S^3$
$-175765\cdot0S^2-275707\cdot3S-120832\cdot8$

$T_{21}(S) = 4\cdot6S^7+262\cdot8S^6+4150\cdot6S^5+14351\cdot3S^4+36379\cdot9S^3$
$+137523\cdot8S^2+233691\cdot8S+105728\cdot7$

$T_{12}(S) = S\cdot6S^7+299\cdot3S^6+6184\cdot7S^5+22998\cdot3S^4+38672\cdot3S^3$
$+106422\cdot8S^2+191676\cdot4S+90624\cdot6$

$$T_{22}(S) = -1\cdot 85^7 - 237\cdot 1S^6 - 6116\cdot 2S^5 - 22123\cdot 3S^4 - 26739\cdot 9S^3 - 69280\cdot 1S^2 - 149660\cdot 9S - 75520\cdot 5$$

If p = 4 and λ_1, λ_2, λ_3 and λ_4 are used as dominant eigenvalues, then the simplified denominator polynomial is

$$\begin{aligned}\Delta_4(S) &= (S+0\cdot 69302487)(S+0\cdot 86648517)(S+1\cdot 2965031+j3\cdot 49839115)\\ &\quad (S+1\cdot 2965031-j3\cdot 49839115)\\ &= S^4+4\cdot 1525171S^3+18\cdot 563875S^2+23\cdot 264936S\\ &\quad +8\cdot 3586945 \end{aligned} \tag{81}$$

The rank of [T(S)] is 16; the ratio of the rank and the dimension is $\frac{16}{2} = 8 = K$, an integer. As expected, 2K = 16 matrix quotients were yielded. The first four matrix quotients obtained after applying Eq. (37) to Eq. (80) are

$$H_1 = \begin{bmatrix} 5.0 & 6.0 \\ 7.0 & 8.0 \end{bmatrix} \quad H_2 = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 25 & 20 \\ 4 & 1 \end{bmatrix} \quad H_4 = \begin{bmatrix} -2.0 & 2.0 \\ 2.5 & -1.0 \end{bmatrix} \tag{82}$$

Substituting the $A_{1,j}$ and H_i in Eq. (81) and (82) into Eq. (48) yields $A_{2,j}$. The reduced models are

$$T(S) = \frac{1}{\Delta(S)}\left[\begin{pmatrix} -4.9858528 & 7.9484098 \\ 5.1678608 & -8.4619989 \end{pmatrix} S^3 + \begin{pmatrix} 3.4797767 & -5.4656713 \\ -5.4605697 & 7.1425118 \end{pmatrix} S^2 + \begin{pmatrix} 42.907577 & 28.001335 \\ 35.454456 & -20.54822 \end{pmatrix} S + \begin{pmatrix} -33.434778 & 25.076084 \\ 29.255431 & -20.896736 \end{pmatrix}\right] \tag{83}$$

Applying a unit-step input to the original system as in the previous example yields

$$T(S) = \begin{bmatrix} Y_1(S) \\ \\ Y_2(S) \end{bmatrix} = \frac{1}{\Delta(S)} \begin{bmatrix} -0\cdot7S^7+38\cdot58S^6+1992\cdot5S^5+6692\cdot2S^4 \\ -11935\cdot1S^3-69342\cdot2S^2-84030\cdot9S-30208\cdot2 \\ 2\cdot8S^7+25\cdot7S^6-1965\cdot6S^5-7772\cdot0S^4 \\ 9640\cdot1S^3+68243\cdot7S^2+84030\cdot9S+30208\cdot2 \end{bmatrix}$$

$$\text{where} \quad \Delta(S) = S^9+678S^8+1285\cdot4S^7+8976\cdot1S^6+38697\cdot4S^5 \\ +105846\cdot1S^4+159414\cdot8S^3+114239\cdot1S^2+30208\cdot2S \tag{84}$$

With the same input, the approximated system outputs are

$$T_4(S) = \begin{bmatrix} Y_1^*(S) \\ \\ Y_2^*(S) \end{bmatrix} = \frac{1}{\Delta_4(S)} \begin{bmatrix} 2\cdot962557S^3-1\cdot9858946S^2-14\cdot906242S \\ -8\cdot358694 \\ -3\cdot2941381S^3+1\cdot6819421S^2+14\cdot906241S \\ +8\cdot358695 \end{bmatrix}$$

$$\text{where} \quad \Delta_4(S) = S^5+4\cdot152517S^4+18\cdot563875S^3+23\cdot264936S^2 \\ +8\cdot3586945 \tag{85}$$

The unit-step responses of the original and approximated systems are shown in Figs. 9 and 10. The corresponding I.S.V. are:

	I.S.V.	
First curve of original system	2·491544	
Second curve of original system	2·976267	
First curve of fourth-order model	3·098814	
Second curve of fourth-order model	3·573341	(86)

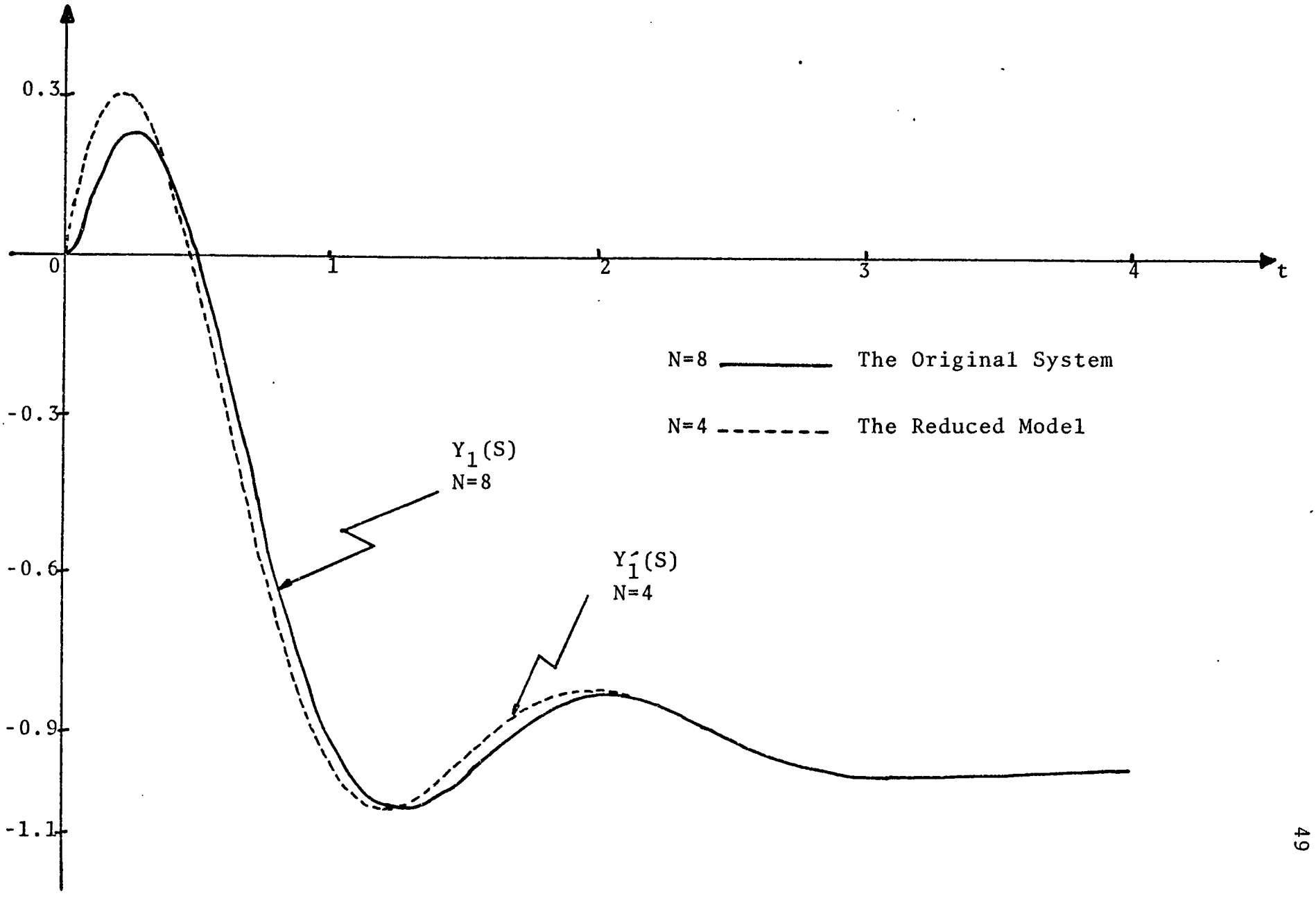

Figure 9. Unit Step Response for Example 6, First Curve.

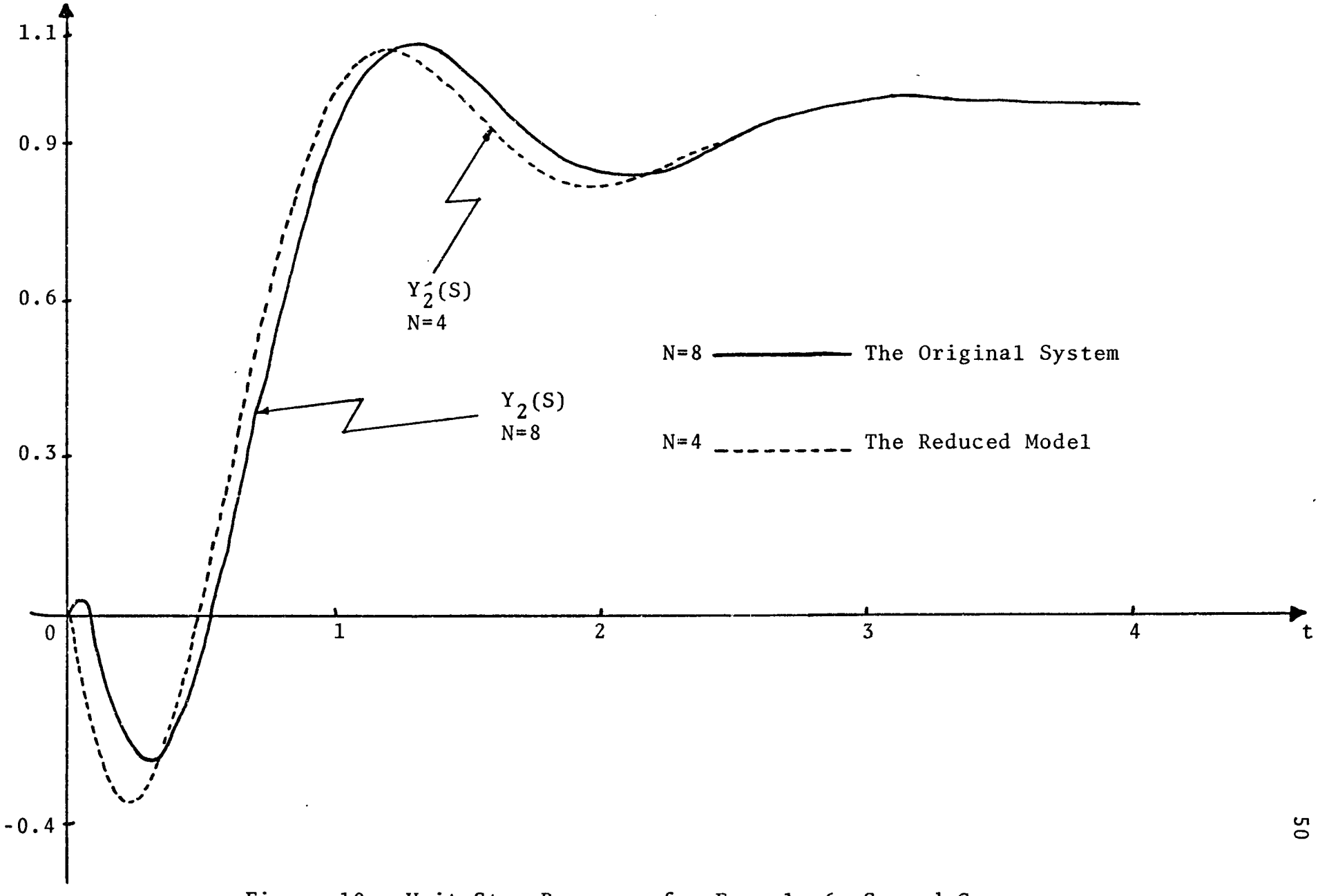

Figure 10. Unit Step Response for Example 6, Second Curve.

Note that if m, the dimension of [T(S)], is an even number and the methods of Chen[5] and Shieh[6] are applied, reduced models of odd degree, p, do not exist. The proposed mixed method[7] can overcome this disadvantage.

CHAPTER III

SYSTEM REDUCTION WITH UNEQUAL NUMBERS OF INPUTS AND OUTPUTS

The model-reduction algorithms developed in the previous chapters deal only with multivariable systems having an equal number of inputs and outputs, and having a transfer-function matrix with no ill-conditioned numerical elements. However, in general, the transfer-function matrix of a practical system is not a square transfer-function matrix and often contains ill-conditioned constants. Under these circumstances, the model reduction by either continued fractions or mixed method would fail. To overcome these deficiencies, an effective method is developed in this chapter for the simplification of multivariable systems with an unequal number of inputs and outputs.

Since the proposed methods depend heavily upon the dimensions of transfer-function matrices, it is convenient to present the approach by the following case studies.[15]

Section 1. Case for the Number of Outputs Less Than the Number of Inputs ($\ell<m$)

Let a multivariable system with m inputs and ℓ outputs be described by the matrix equation

$$[Y_o(S)] = [G_o(S)][U_o(S)] \tag{87}$$

and the transfer-function matrix

$$[G_o(S)] = \frac{1}{\Delta_o(S)}[Q(S)] \tag{88}$$

where

$$\Delta_o(S) = \sum_{i=1}^{n+1} a_i S^{i-1} = \prod_{i=1}^{n} (S-\lambda_i), a_1 \neq 0, a_{n+1} = 1,$$

$$\lambda_i \neq 0$$

and

$$[Q(S)] = \sum_{i=1}^{n} Q_i S^{i-1} = [Q_1 S^{n-1} + Q_2 S^{n-2} + \ldots + Q_{n-1} S + Q_n]$$

Consider the rank of $[G_o(S)] = r_o$, which is required to modify the rectangular matrix $[G_o(S)]$ and to construct a new square transfer-function matrix $[T_o(S)]$ with rank $r = Kxm$. The matrix $[T_o(S)]$ can be obtained by adding another square matrix $[G_2(S)]$ whose rank is $r-r_o$ to the modified $[G_o(S)]$. The modified system is:

$$[Y(S)] = [T_o(S)][U_o(S)] \tag{89}$$

where

$$[Y(S)] = \begin{bmatrix} (\ell x1) \\ Y_o(S) \\ \hline ((m-\ell)x1) \\ Y_1(S) \end{bmatrix}$$

$$[T_o(S)] = [G_1(S)]+[G_2(S)]$$

$$[G_1(S)] = \begin{bmatrix} (\ell x m) \\ G_o(S) \\ \hline ((m-1)xm) \\ 0 \end{bmatrix} = \frac{1}{\Delta_o(S)} \begin{bmatrix} (\ell x m) \\ Q(S) \\ \hline ((m- \ )xm) \\ 0 \end{bmatrix}$$

$$[G_2(S)] = \begin{bmatrix} (\ell x m) \\ 0 \\ \hline ((m-\ell)xm) \\ R(S) \end{bmatrix} = \frac{1}{\Delta_o(S)} \begin{bmatrix} (\ell x m) \\ 0 \\ \hline ((m-\ell)xm) \\ R \end{bmatrix}$$

$$[T_o(S)] = \frac{1}{\Delta_o(S)} \begin{bmatrix} (\ell x m) \\ Q(S) \\ \hline ((m-\ell)xm) \\ R \end{bmatrix}$$

The elements in the constant matrix [R] should be chosen so that the rank of $[T_o(S)]$ = Kxm where K is an integer. Since $[T_o(S)]$ is a square matrix with rank $[T_o(S)]$ = Kxm, the methods proposed in Chapter II can be applied to obtain the reduced model. The reduced model for the modified system $[T_o(S)]$ is

$$[Y_d(S)] = [T_d(S)][U_o(S)] \tag{90}$$

where

$$[Y_d(S)] = \begin{bmatrix} (\ell x 1) \\ Y_{d1}(S) \\ \hline ((m-\ell)x1) \\ Y_{d2}(S) \end{bmatrix}$$

and

$$[T_d(S)] = \begin{bmatrix} (\ell x m) \\ T_{d1}(S) \\ \hline ((m-\ell)xm) \\ T_{d2}(S) \end{bmatrix}$$

The reduced model for the original system $[G_o(S)]$ can be obtained by partitioning the matrix in Eq. (90), or

$$[Y_{d1}(S)] = [T_{d1}(S)][U_o(S)]$$
$$\simeq [Y_o(S)] = [G_o(S)][U_o(S)] \tag{91}$$

Example 7

To illustrate the techniques stated above, consider the following transfer-function matrix $[G_o(S)]$ with $\ell = 1$ and $m = 2$

$$[G_o(S)] = \frac{[S+1.5, 4]}{(S+1)(S+2)(S+100)} \tag{92}$$

The characteristic polynomial of the matrix is

$$\Delta(S) = (S+1)(S+2)(S+100)$$
$$= S^3+103S^2+302S+200$$

These procedures are performed to obtain a new modified transfer-function matrix:

$$[T_o(S)] = [G_1(S)]+[G_2(S)]$$
$$= \frac{\begin{bmatrix} S+1.5 & 4 \\ 0 & 0 \end{bmatrix}}{S^3+103S^2+302S+200} + \frac{\begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}}{S^3+103S^2+302S+200}$$
$$\frac{\begin{bmatrix} S+1.5 & 4 \\ 0.1 & 0 \end{bmatrix}}{S^3+103S^2+302S+200} \tag{93}$$

The rank of $[T_o(S)]$ is 6. The ratio of the rank and the dimension of this modified system is $\frac{6}{2} = 3 = K$, an integer. Using Eq. (37), we have $2K = 6$ matrix quotients:

$$H_1 = \begin{bmatrix} 0.0 & 0.2\text{x}10^4 \\ 0.5\text{x}10^2 & -0.75\text{x}10^3 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.71597737\text{x}10^{-2} & 0.13245\text{x}10^{-1} \\ 0.33112583\text{x}10^{-3} & 0.0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0.0 & -0.885475\text{x}10^4 \\ -0.221368\text{x}10^3 & -0.23805\text{x}10^3 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 0.239697\text{x}10^{-2} & -0.136331\text{x}10^{-1} \\ -0.34082797\text{x}10^{-3} & 0.0 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} -0.355512\text{x}10^{-12} & 0.10314005\text{x}10^8 \\ 0.25785012\text{x}10^6 & 0.25381463\text{x}10^9 \end{bmatrix}$$

$$H_6 = \begin{bmatrix} -0.95567532\text{x}10^{-2} & 0.38808577\text{x}10^{-3} \\ 0.97021442\text{x}10^{-5} & 0.0 \end{bmatrix} \tag{94}$$

If the mixed method is applied and H_1, H_2 are used, the reduced model for this modified $[T_o(S)]$ is

$$[T_d(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.00985S+0.015 & -0.0004S+0.04 \\ -0.00001S+0.001 & 0.0 \end{bmatrix} \tag{95}$$

where $\Delta_2(S) = (S+1)(S+2)$

$= S^2+3S+2$

and the reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{[0.00985S+0.015,\ -0.0004S+0.04]}{S^2+3S+2} \tag{96}$$

In examining the effect of the value of (m-1)xm matrix R in Eq. (89), consider the transfer-function matrix in Eq. (92)

$$[G_o(S)] = \frac{[S+1.5,\ 4]}{(S+1)(S+2)(S+100)} \tag{97}$$

and obtain another new modified transfer-function matrix as:

$$[T_o'(S)] = [G_1(S)]+[G_2'(S)]$$

$$= \frac{\begin{bmatrix} S+1.5 & 4 \\ 0 & 0 \end{bmatrix}}{S^3+103S^2+302S+200} + \frac{\begin{bmatrix} 0 & 0 \\ 100 & 0 \end{bmatrix}}{S^3+103S^2+302S+200}$$

$$= \frac{\begin{bmatrix} S+1.5 & 4 \\ 100 & 0 \end{bmatrix}}{S^3+103S^2+302S+200} \tag{98}$$

The rank of $T_o'(S)$ is also 6, $K = \frac{r}{m} = \frac{6}{2} = 3$; hence six matrix quotients are expected:

$$H_1 = \begin{bmatrix} 0.0 & 0.2\text{x}10 \\ 0.5\text{x}10^2 & -0.75 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.71597737\text{x}10^{-2} & 0.13245033\text{x}10^{-1} \\ 0.33112583 & 0.0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0.0 & -0.88547573\text{x}10 \\ -0.2213689\text{x}10^3 & -0.23805024 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 0.23969795\text{x}10^{-2} & -0.13633119\text{x}10^{-1} \\ -0.34082797 & 0.0 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} 0.0 & 0.10314005\text{x}10^5 \\ 0.25785012\text{x}10^6 & 0.25381463\text{x}10^6 \end{bmatrix}$$

$$H_6 = \begin{bmatrix} -0.95567532\text{x}10^{-2} & 0.38808577\text{x}10^{-3} \\ 0.97021442\text{x}10^{-2} & 0.0 \end{bmatrix} \tag{99}$$

Again the mixed method is applied and H_1, H_2 are used. The reduced model for $[T_o'(S)]$ is

$$[T_d'(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.00985S+0.015 & -0.0004S+0.04 \\ -0.01S+1 & 0.0 \end{bmatrix} \quad (100)$$

where $\Delta_2(S) = S^2+3S+2$. The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}'(S)] = \frac{[0.00985S+0.015,\ -0.0004S+0.04]}{S^2+3S+2} \quad (101)$$

Observe that $[T_{d1}(S)] = [T_{d1}'(S)]$; it can be concluded that the values of the $(m-\ell)x\ell$ matrix R do not affect the final values of the reduced model. We can choose a convenient $(m-\ell)x\ell$ R matrix for ease in computation, provided that the rank of $[T_o(S)]$ is Kxm, where K is a proper integer.

Applying a unit-step input to the original system and the reduced model yields

$$G_o(S) = \frac{S+5.5}{S^3+103S^2+302S+200} \cdot \frac{1}{S}$$

$$T_{d1}(S) = \frac{0.00945S+0.055}{S^2+3S+2} \cdot \frac{1}{S} \quad (102)$$

The unit-step responses are shown in Fig. 11; the approximation by this procedure is very satisfactory. The integral square value of the original system is $0.26821228x10^{-3}$ and that of the reduced model is $0.26696687x10^{-3}$. Note that the integral square value of the original system is very close to that of the reduced model.

Example 8

The power of the mixed method and the modified matrix-continued fraction approximations for a multivariable

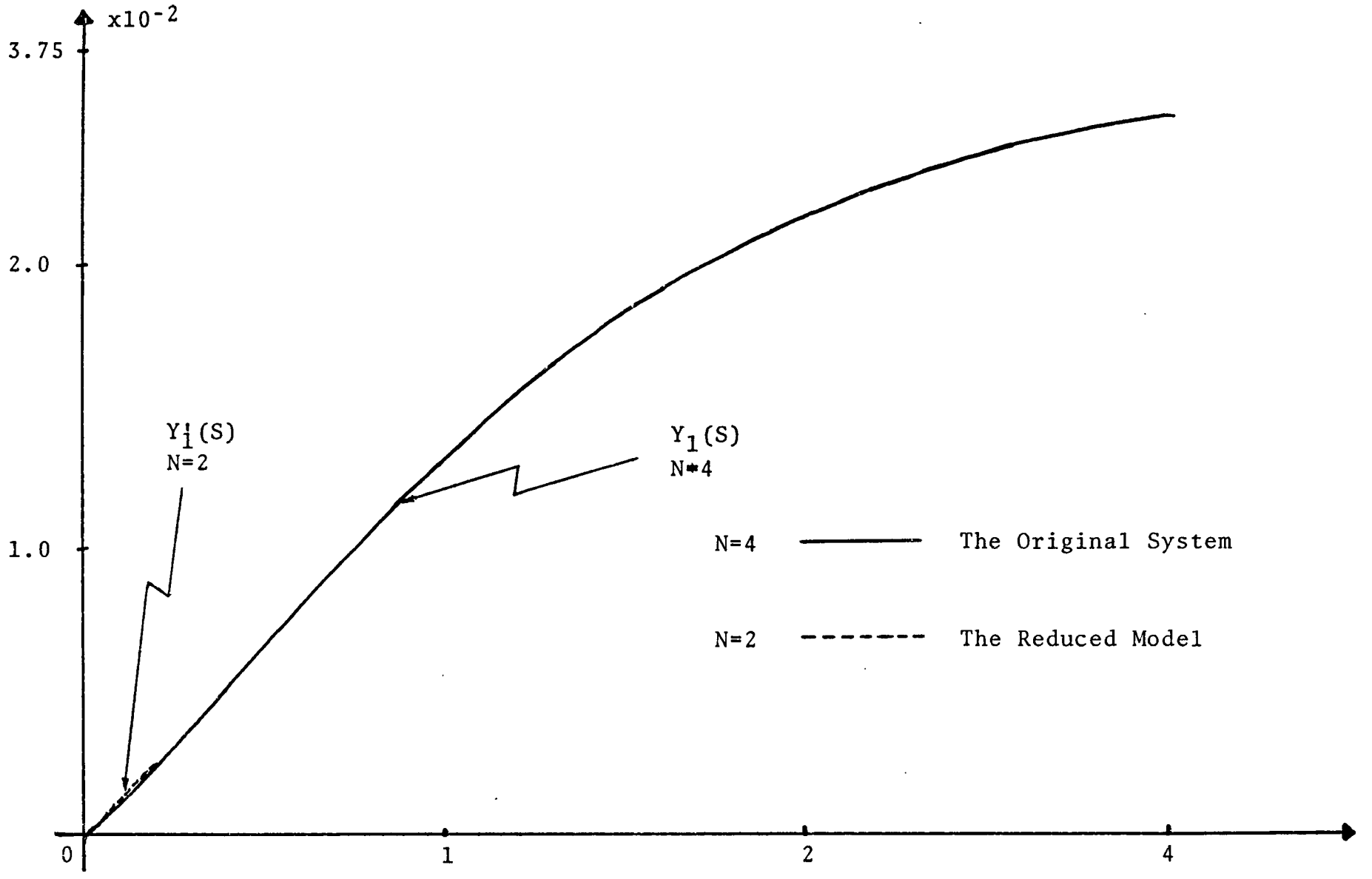

Figure 11. Unit Step Response for Example 7.

systems where the number of inputs exceed the number of outputs may be illustrated by considering this following transfer-function matrix $[G_o(S)]$ with m = 3 and ℓ = 2.

$$[G_o(S)] = \frac{1}{\Delta_o(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) & G_{13}(S) \\ G_{21}(S) & G_{22}(S) & G_{23}(S) \end{bmatrix} \tag{103}$$

where

$$\begin{aligned}
\Delta_o(S) &= S^8+30.41S^7+358.4295S^6+2913.8638S^5+18110.567S^4+ \\
&\quad 67556.983S^3+173383.58S^2+149172.19S+37752.826 \\
&= (S+0.35+j6.8)(S+0.35-j6.8)(S+0.46) \\
&\quad (S+0.75)(S+2.2+j3.6)(S+2.2-j3.6)(S+8.5) \\
&\quad (S+15.6) \\
G_{11}(S) &= 19.82S^7+429.252S^6+4843.8072S^5+45575.952S^4 \\
&\quad +241544.69S^3+905812.05S^2+1890443.1S \\
&\quad +842597.95 \\
G_{12}(S) &= 6.6S^7+157.749S^6+3039.363S^5+15736.191S^4 \\
&\quad +89601.204S^3+317009.53S^2+732817.47S \\
&\quad +312000.5 \\
G_{13}(S) &= 23.6S^7+651.76S^6+8867.5939S^5+62029.838S^4 \\
&\quad +336313.03S^3+1316700.5S^2+2987484.2S \\
&\quad +1671748.9 \\
G_{21}(S) &= 2.96S^7+65.31S^6+828.7689S^5+6956.6746S^4 \\
&\quad +33445.715S^3+111211.67S^2+136814.3S \\
&\quad +33487.533 \\
G_{22}(S) &= 15.8S^7+397.818S^6+3871.993S^5+30696.33S^4 \\
&\quad +140696.69S^3+475842.89S^2+937588.71S \\
&\quad +405193.11
\end{aligned}$$

$$G_{23}(S) = 4.4S^7+101.161S^6+1404.1517S^5+10855.522S^4 +65863.554S^3+236964.75S^2+495290.54S +204527.34$$

The characteristic polynomial of the matrix $[G_o(S)]$ is $\Delta(S) = \{\Delta_o(S)\}^2$. The following procedures are performed to obtain a new modified transfer-function matrix:

$$[T_o(S)] = [G_1(S)]+[G_2(S)]$$

$$= \frac{1}{\Delta_o(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) & G_{13}(S) \\ G_{21}(S) & G_{22}(S) & G_{23}(S) \\ \hline 1. & 0. & 0. \end{bmatrix} \qquad (104)$$

The rank of $[T_o(S)]$ is 24. The ratio of the rank to the dimension of $[T_o(S)]$ is $\frac{24}{3} = 8 = K$, an integer. By using Eq. (37), we have $2K = 16$ matrix quotients. The First four quotients are:

$$[H_1] = \begin{bmatrix} 0. & 0. & 0.37752826\text{x}10^5 \\ -0.12584553\text{x}10^{-1} & 0.10286259 & 0.71591038\text{x}10^4 \\ 0.24931503\text{x}10^{-1} & -0.19197368\text{x}10^{-1} & -0.20364361\text{x}10^5 \end{bmatrix}$$

$$[H_2] = \begin{bmatrix} 0.11480738\text{x}10^2 & 0.51266112\text{x}10 & 0.20667866\text{x}10^2 \\ 0.83229395 & 0.65556553\text{x}10 & 0.34686103\text{x}10 \\ 0.67036624\text{x}10^{-5} & 0. & 0. \end{bmatrix}$$

$$[H_3] = \begin{bmatrix} -0.9476532\text{x}10^{-16} & 0.6317688\text{x}10^{-6} & -0.12834169\text{x}10^6 \\ -0.79448967\text{x}10 & 0.49349865\text{x}10 & -0.19437105\text{x}10^7 \\ 0.15729226\text{x}10^2 & -0.84312433\text{x}10 & 0.39432597\text{x}10^7 \end{bmatrix}$$

$$[H_4] = \begin{bmatrix} 0.32055347\text{x}10 & 0.93388214 & 0.47820585 \\ 0.13836121\text{x}10 & 0.16462128\text{x}10 & 0.85234565 \\ -0.10084175\text{x}10^{-4} & -0.21698973\text{x}10^{-21} & -0.89141847\text{x}10^{-22} \end{bmatrix}$$

If the method of the matrix-continued fraction is applied and H_1 and H_2 are used, then (by applying Eq. (48) or Eq. (55)) the reduced model for the modified system $[T_o(S)]$ is

$$[(T_d)] = \frac{1}{\Delta(S)} \begin{bmatrix} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \\ \hdashline T_{31}(S) & T_{32}(S) & T_{33}(S) \end{bmatrix} \tag{106}$$

where

$\Delta(S)$ = The characteristic polynomial of the reduced model

$= S^3+1.3115805S^2+0.54131967S+0.069200278$

$= (S+0.2530821524)(S+0.4475236322)(S+0.6109847154$

$T_{11}(S) = 0.11480738x10^2S^2+0.94441257x10S+0.15444675x10$

$T_{12}(S) = 0.51266112x10S^2+0.35571607x10S+0.57189153$

$T_{13}(S) = 0.20667866x10^2S^2+0.17338541x10^2S+0.30642869x10$

$T_{21}(S) = 0.83229395S^2+0.48840185S+0.6138207x10^{-1}$

$T_{22}(S) = 0.65556553x10S^2+0.45937866x10S+0.74271197$

$T_{23}(S) = 0.34686103x10S^2+0.2359161x10S+0.37489508$

$T_{31}(S) = 0.67036624x10^{-5}S^2+0.70958826x10^{-5}S$
$+0.18329827x10^{-5}$

$T_{32}(S) = 0.$

$T_{33}(S) = 0.$

The reduced model for the original system $[G_o(S)]$ is

$$T_{d1}(S) = \frac{1}{\Delta(S)} \begin{bmatrix} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \end{bmatrix} \tag{107}$$

The zeros of $\Delta(S) = 0$ in Eq. (107) are the equivalent dominant poles of the original system $[G_o(S)]$.

If the mixed method is applied and H_i, $i = 1,\ldots, 4$ are used, then the reduced model for the modified system $[T_o(S)]$ is

$$[T_d(S)] = \frac{1}{\Delta_4(S)} \begin{bmatrix} T_{11}^*(S) & T_{12}^*(S) & T_{13}^*(S) \\ T_{21}^*(S) & T_{22}^*(S) & T_{23}^*(S) \\ \hline T_{31}^*(S) & T_{32}^*(S) & T_{33}^*(S) \end{bmatrix} \tag{108}$$

where $\Delta_4(S)$ = The least common-denominator polynomial

$$= (S+0.35+j6.8)(S+0.35-j6.8)(S+0.46)(S+0.75)$$

$$= S^4+1.91S^3+47.5545S^2+56.340125S+15.9950625$$

$$T_{11}^*(S) = -0.11487882\text{x}10S^3+0.67114913\text{x}10^2S^2 +0.64781099\text{x}10^3S+0.35699068\text{x}10^3$$

$$T_{12}^*(S) = 0.40355433\text{x}10S^3+0.11092337\text{x}10^2S^2 +0.25377828\text{x}10^3S+0.13218792\text{x}10^3$$

$$T_{13}^*(S) = 0.13009015S^3+0.6829685\text{x}10^2S^2+0.96192116\text{x}10^3S +0.70828415\text{x}10^3$$

$$T_{21}^*(S) = -0.16371447\text{x}10S^3+0.23321667\text{x}10^2S^2 +0.51879505\text{x}10^2S+0.14187951\text{x}10^2$$

$$T_{22}^*(S) = 0.34420703\text{x}10S^3+0.44147724\text{x}10^2S^2 +0.32359929\text{x}10^3S+0.17167163\text{x}10^3$$

$$T_{23}^*(S) = 0.842381165^3+0.16914937\text{x}10^2S^2 +0.17267462\text{x}10^3S+0.86653849\text{x}10^2$$

$$T_{31}^*(S) = 0.93739989\text{x}10^{-6}S^3+0.31920638\text{x}10^{-4}S^2 -0.18173293\text{x}10^{-3}S+0.4236785\text{x}10^{-3}$$

$T^*_{32}(S) = 0.$

$T^*_{33}(S) = 0.$

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{1}{\Delta_4(S)} \begin{bmatrix} T^*_{11}(S) & T^*_{12}(S) & T^*_{13}(S) \\ T^*_{21}(S) & T^*_{22}(S) & T^*_{23}(S) \end{bmatrix} \qquad (109)$$

The characteristic polynomial of $[T_{d1}(S)]$ is $\Delta(S) = \{\Delta_4(S)\}^2$. Note that the reduced model retains the dominant poles of the original system. The unit-step response curves of the original system and the reduced models in Eq. (103), (107), and (109) are compared in Figs. 12 and 13. The reduced models are stable and have a good approximation in steady-state responses but a slight discrepancy in the transient response.

The I.S.V. of this system are stated below:

	I.S.V.
First curve of original system	0.166396557×10^4
Second curve of original system	0.125646808×10^3
First curve of the fourth order (mixed method) reduced model	0.256096679×10^4
Second curve of the fourth order (mixed method) reduced model	0.23946464×10^3

First curve of the third order (second Caver form) reduced model 0.140070385×10^4

Second curve of the third order (second Caver form) reduced model 0.1100467834×10^3 (110)

Observe that the second Caver form approximation is better than that of mixed method in this particular example.

Example 9

For another illustration, consider the following transfer-function matrix $[G_o(S)]$ with m = 3 and ℓ = 2

$$[G_o(S)] = \frac{1}{\Delta_o(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) & G_{13}(S) \\ G_{21}(S) & G_{22}(S) & G_{23}(S) \end{bmatrix} \tag{111}$$

where

$$\begin{aligned} \Delta_o(S) &= S^8+86.487S^7+2587.496S^6+32845.65S^5+171583.354S^4 \\ &\quad +313189.964S^3+245116.102S^2+83400.489S \\ &\quad +11309.76782 \\ &= (S+0.2996+j0.1655)(S+0.2996-j0.1655) \\ &\quad (S+0.9776+j0.1661)(S+0.9776-j0.1661) \\ &\quad (S+7.448)(S+14.998)(S+22.61)(S+38.888) \end{aligned}$$

$$\begin{aligned} G_{11}(S) &= -3.217S^7+19.627S^6+1146.549S^5+ 3716.882S^4 \\ &\quad +223.874S^3-1729.049S^2-40283.116S+6280.031 \end{aligned}$$

$$\begin{aligned} G_{12}(S) &= -1.693S^7+8.993S^6+934.667S^5+1079.064S^4 \\ &\quad -701.762S^3-34910.78S^2+4858.702S-8803.412 \end{aligned}$$

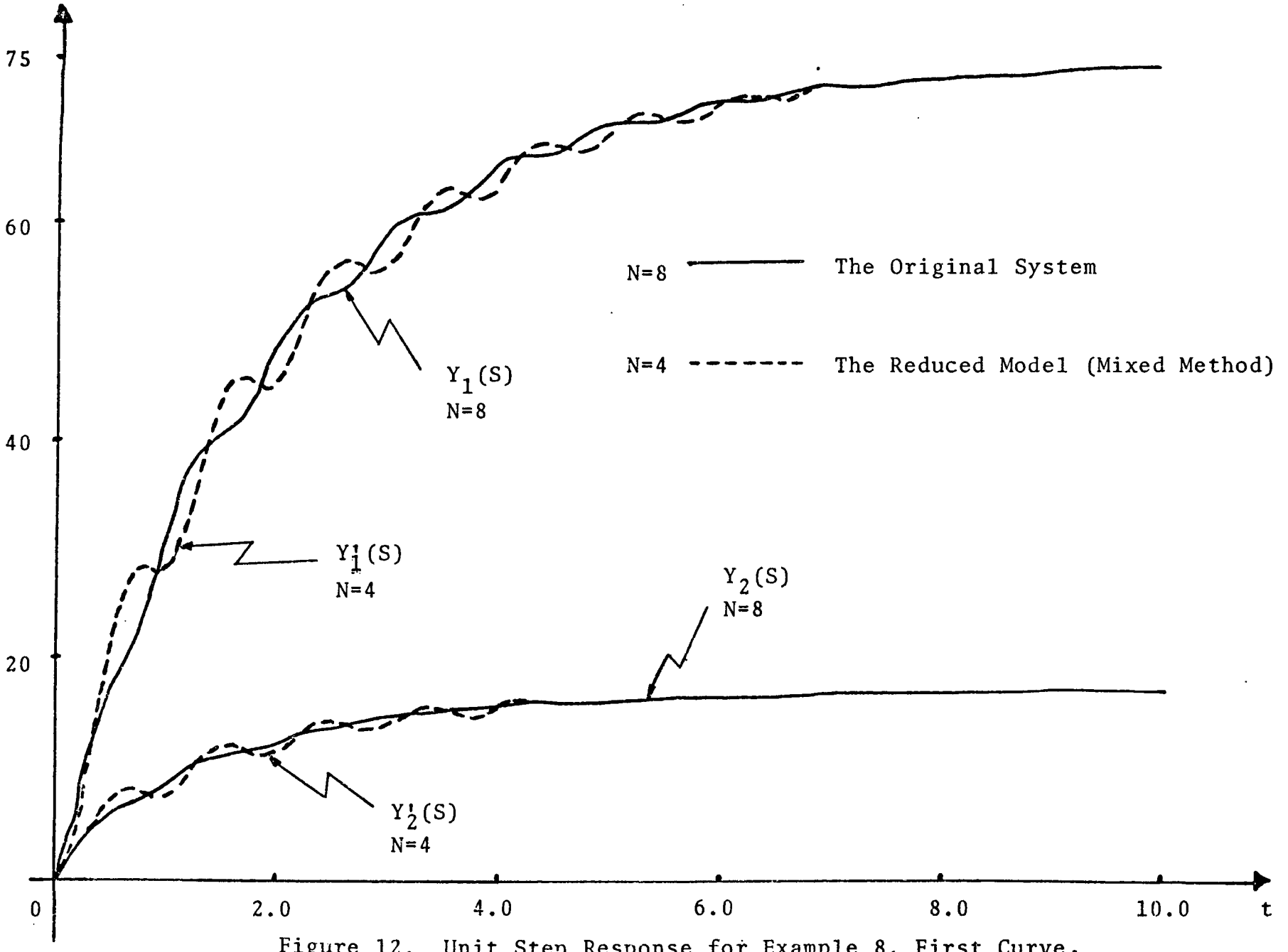

Figure 12. Unit Step Response for Example 8, First Curve.

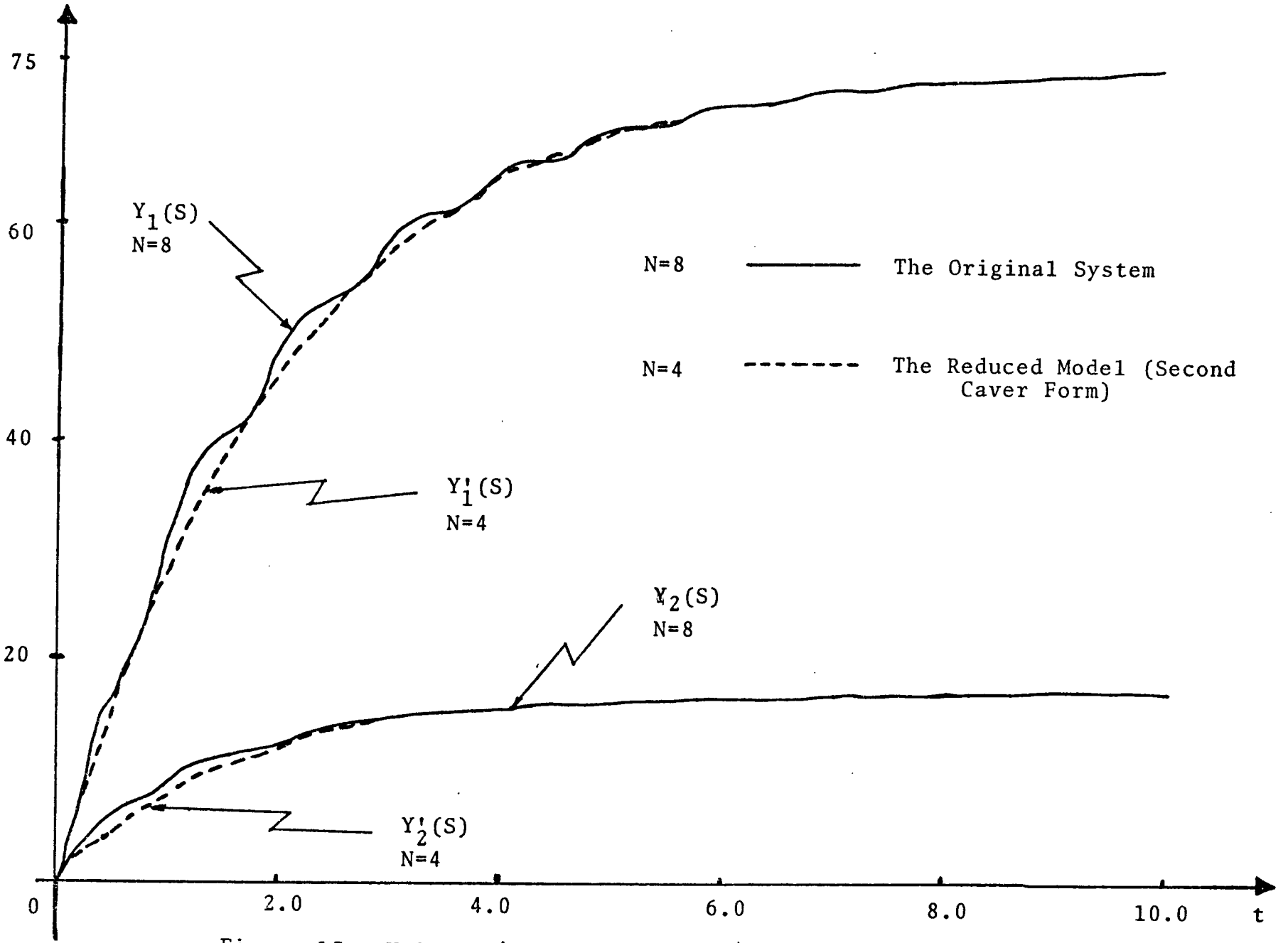

Figure 13. Unit Step Response for Example 8, Second Curve.

$$G_{13}(S) = 3.465S^7+11.861S^6+233.666S^5+3896.727S^4 -876.014S^3-41992.207S^2-31967.676S -7900.486$$

$$G_{21}(S) = 7.632S^7+12.031S^6-501.74S^5-6360.227S^4 +9070.185S^3+36795.67S^2+160754.791S +4797.452$$

$$G_{22}(S) = -2.951S^7+7.313S^6-882.113S^5-769.914S^4 +4416.643S^3+2416.532S^2+29642.495S +2497.208$$

$$G_{23}(S) = -1.212S^7+13.305S^6-692.829S^5-1872.906S^4 -2846.545S^3+34630.529S^2+342405.841S +5335.73$$

The characteristic polynomial of the matrix $[G_o(S)]$ is $\Delta(S) = \{\Delta_o(S)\}^2$. Obtain the new modified transfer-function matrix

$$[T_o(S)] = [G_1(S)]+[G_2(S)] \tag{112}$$

$$= \frac{1}{\Delta(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) & G_{13}(S) \\ G_{21}(S) & G_{22}(S) & G_{23}(S) \\ \hline 1.0 & 0. & 0. \end{bmatrix}$$

The rank of $[T_o(S)]$ is 24, hence the ratio of the rank to the dimension is $\frac{24}{3} = 8 = K$, an integer. Again applying Eq. (37), the first four of the $2K = 16$ matrix quotients are:

$$[H_1] = \begin{bmatrix} 0.38318984\text{x}10^{-16} & 0.38318984\text{x}10^{-16} & 0.11309768\text{x}10^{5} \\ -0.22150578\text{x}10 & -0.32797824\text{x}10 & 0.29645231\text{x}10^{5} \\ 0.10366829\text{x}10 & 0.36546202\text{x}10 & -0.24043266\text{x}10^{5} \end{bmatrix}$$

$$[H_2] = \begin{bmatrix} 0.45116321\text{x}10^{-1} & -0.92898212\text{x}10^{-1} & -0.79173271\text{x}10^{-1} \\ 0.45319024\text{x}10^{-1} & 0.22768179\text{x}10^{-1} & 0.16408817\text{x}10^{-1} \\ 0.11990337\text{x}10^{-4} & 0.18578757\text{x}10^{-22} & -0.10846143\text{x}10^{-21} \end{bmatrix}$$

$$[H_3] = \begin{bmatrix} -0.30038726\text{x}10^{-15} & -0.5362692\text{x}10^{-16} & -0.28376926\text{x}10^{5} \\ 0.45986809\text{x}10 & 0.33347733\text{x}10 & -0.59570614\text{x}10^{5} \\ -0.11817562\text{x}10 & -0.3404811\text{x}10 & 0.33386715\text{x}10^{5} \end{bmatrix}$$

$$[H_4] = \begin{bmatrix} -0.2560898 & 0.10830943 & -0.71889798\text{x}10^{-1} \\ 0.63462792 & 0.87967091\text{x}10^{-1} & 0.139994\text{x}10 \\ -0.21212193\text{x}10^{-4} & -0.15744114\text{x}10^{-21} & -0.15088878\text{x}10^{-21} \end{bmatrix} \tag{113}$$

Using the mixed method with H_i, $i = 1,\ldots, 4$, the reduced model for the modified system $[T_o(S)]$ is

$$[T_d(S)] = \frac{1}{\Delta_4(S)} \begin{bmatrix} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \\ \hline T_{31}(S) & T_{32}(S) & T_{33}(S) \end{bmatrix} \tag{114}$$

where $\Delta_4(S) = (S+0.2996+j0.1655)(S+0.2996-j0.1655)$

$(S+0.9776+j0.1661)(S+0.9776-j0.1661)$

$= S^4+2.5545S^3+2.2721S^2+0.81822S+0.11519$

$T_{11}(S) = -0.14279847\text{x}10^{-1}S^3+0.96772826\text{x}10^{-1}S^2$

$-0.42761604S+0.63962127\text{x}10^{-1}$

$$T_{12}(S) = 0.92769451\times10^{-1}S^3-0.37346816S^2$$
$$+0.73782622\times10^{-1}S-0.89662764\times10^{-1}$$
$$T_{13}(S) = 0.91633318\times10^{-1}S^3-0.34349318S^2-0.30378618S$$
$$-0.80466461\times10^{-1}$$
$$T_{21}(S) = 0.72275578\times10^{-1}S^3-0.66460835\times10^{-1}S^2$$
$$+0.16240473\times10S+0.48862054\times10^{-1}$$
$$T_{22}(S) = 0.53136006\times10^{-1}S^3-0.55925154\times10^{-1}S^2$$
$$+0.29501683S+0.25434067\times10^{-1}$$
$$T_{23}(S) = 0.49534084\times10^{-1}S^3-0.58958648S^2$$
$$+0.34726783\times10S+0.54344417\times10^{-1}$$
$$T_{31}(S) = -0.12341126\times10^{-6}S^3+0.51027315\times10^{-6}S^2$$
$$-0.27599242\times10^{-5}S+0.10185001\times10^{-4}$$
$$T_{32}(S) = 0.87362005\times10^{-19}S^3-0.28584778\times10^{-20}S^2$$
$$-0.57618962\times10^{-20}S+0.3902789\times10^{-21}$$
$$T_{33}(S) = 0.76839590\times10^{-19}S^3+0.43080259\times10^{-19}S^2$$
$$-0.2555777\times10^{-19}S+0.0$$

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{1}{\Delta_4(S)}\begin{bmatrix} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \end{bmatrix} \tag{115}$$

The characteristic polynomial of $[T_{d1}(S)]$ is $\Delta(S) = \{\Delta_4(S)\}^2$.

The unit-step response curves of the original system and the reduced model in Eqs. (111) and (115) are compared in Figs. 14 and 15. The approximation for this multivariable system is very satisfactory. The model-reduction by the

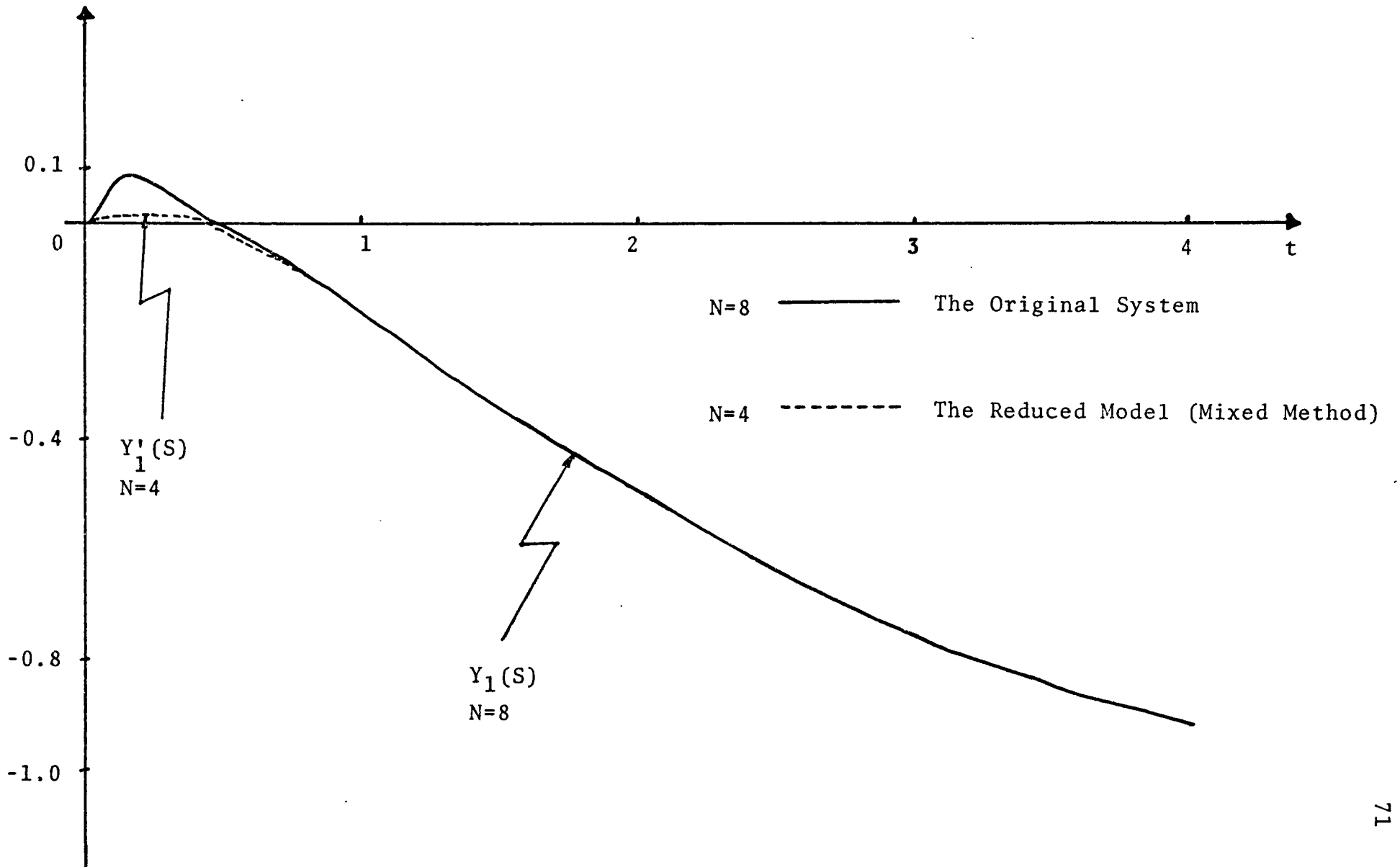

Figure 14. Unit Step Response for Example 9, First Curve.

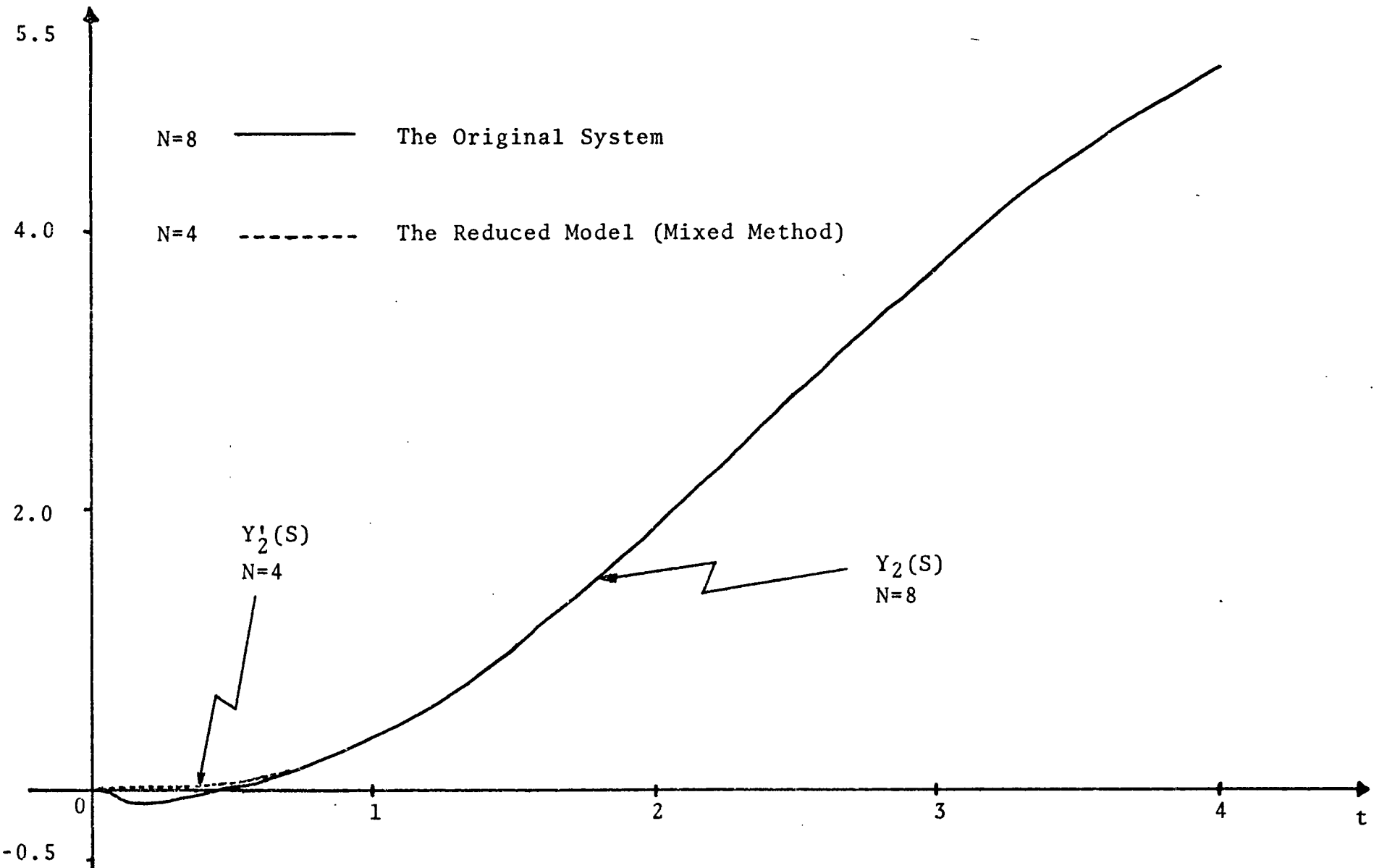

Figure 15. Unit Step Response for Example 9, Second Curve.

matrix-continued fraction for this particular system is not given, because its third order approximation is unstable and the sixth order approximation is far from satisfactory.

The numerical comparison by the I.S.V. is:

	I.S.V.
First curve of original system	0.3814428
Second curve of original system	0.113688755×10^2
First curve of the reduced model	0.27033925
Second curve of the reduced model	0.111731157×10^2

(116)

Section 21 Case for the Number of Outputs Greater Than the Number of Inputs ($\ell > m$)

Restating the m inputs and 1 outputs multivariable system:

$$[Y_o(S)] = [G_o(S)][U_o(S)] \tag{117}$$

The transfer-function matrix

$$[G_o(S)] = \frac{1}{\Delta_o(S)} [Q(S)] \tag{118}$$

Assume that the rank of $[G_o(S)]$ is r_o. A new square matrix $[T_o(S)]$ with rank $[T_o(S)] = K \times \ell$ where K is an integer is to be constructed by modifying the matrix $[G_o(S)]$ and by adding another matrix $[G_2(S)]$:

$$[T_o(S)] = [G_1(S)]+[G_2(S)] \tag{119}$$

where

$$[G_1(S)] = \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ G_o(S) & 0 \end{array}\right] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ Q(S) & 0 \end{array}\right]$$

and

$$[G_2(S)] = \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ 0 & R(S) \end{array}\right] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ 0 & R \end{array}\right]$$

$$[T_o(S)] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ Q(S) & R \end{array}\right]$$

The elements in the constant matrix [R] should be chosen in such a way that rank of $[T_o(S)] = Kx\ell$, where K is an integer. Applying the proposed procedures in Chapter II yields the reduced model for the modified system $[T_o(S)]$

$$[Y_d(S)] = [T_d(S)][U_d(S)] \tag{120}$$

where

$$[Y_d(S)] = \begin{bmatrix} (\ell x 1) \\ Y_{d1}(S) \end{bmatrix} + \begin{bmatrix} (\ell x 1) \\ Y_{d2} \end{bmatrix}$$

$$[U_d(S)] = \begin{bmatrix} (mx1) \\ U_o(S) \\ \hline ((\ell-m)x1) \\ U_1(S) \end{bmatrix}$$

$$[T_d(S)] = \left[\begin{array}{c|c} (\ell x m) & (\ell x(\ell-m)) \\ T_{d1}(S) & T_{d2}(S) \end{array}\right]$$

The reduced model for the original system $[G_o(S)]$ can be obtained by partitioning the matrix in Eq. (120);

$$[Y_{d1}(S)] = [T_{d1}(S)][U_o(S)] \tag{121}$$
$$\simeq [Y_o(S)] = [G_o(S)][U_o(S)]$$

Example 10

To illustrate these proposed procedures and to examine the effect of the matrix R, consider this simple system with $m = 1$ and $\ell = 2$

$$[G_o(S)] = \frac{\begin{bmatrix} S+1.5 \\ 4 \end{bmatrix}}{(S+1)(S+2)(S+100)} \tag{122}$$

The characteristic polynomial of the matrix $[G_o(S)]$ is $\Delta_o(S) = (S+1)(S+2)(S+100)$. The new modified transfer-function matrix is

$$[T_o(S)] = [G_1(S)]+[G_2(S)]$$
$$= \frac{1}{\Delta_o(S)} \begin{bmatrix} S+1.5 & 0 \\ 4 & 0.1 \end{bmatrix} \tag{123}$$

The rank is 6, and the ratio of rank to dimension is $\frac{6}{2} = 3 = K$, an integer; hence we have a yield of $2K = 6$ matrix quotients. The first two matrix quotients are:

$$H_1 = \begin{bmatrix} 0.13333333\text{x}10^{3} & 0.27755576\text{x}10^{-14} \\ -0.53333333\text{x}10^{4} & 0.2\text{x}10^{4} \end{bmatrix}$$
$$H_2 = \begin{bmatrix} 0.88932806\text{x}10^{-2} & 0.50821977\text{x}10^{-20} \\ 0.13245033\text{x}10^{-1} & 0.33112583\text{x}10^{-3} \end{bmatrix} \tag{124}$$

Applying the mixed method and using H_1, H_2, then the reduced model for the modified system $[T_o(S)]$ is

$$[T_d(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.98499999\text{x}10^{-2}S+0.15\text{x}10^{-1} & 0.11564823\text{x}10^{-19}S-0.135525\text{x}10^{-19} \\ -0.4\text{x}10^{-3}S+0.4\text{x}10^{-1} & -0.99999\text{x}10^{-5}S+0.1\text{x}10^{-2} \end{bmatrix} \quad (125)$$

where $\Delta_2(S)$ = The least common-denominator polynomial

$= (S+1)(S+2)$

$= S^2+3S+2$

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.98499999\text{x}10^{-2}S+0.15\text{x}10^{-1} \\ -0.4\text{x}10^{-3}S+0.4\text{x}10^{-1} \end{bmatrix} \quad (126)$$

The characteristic polynomial of $[T_{d1}(S)]$ is

$\Delta_2(S) = (S+1)(S+2)$.

Now remodel the same system as:

$$[T_o'(S)] = [G_1(S)]+[G_2'(S)]$$

$$= \frac{1}{\Delta_o(S)} \begin{bmatrix} S+1.5 & 0 \\ 4 & 100 \end{bmatrix} \quad (127)$$

The rank is also 6, and the ratio of rank to dimension is $\frac{6}{2} = 3 = K$, an integer; hence we have a yield of $2K = 6$ matrix quotients. The first two matrix quotients are:

$$H_1' = \begin{bmatrix} 0.0 & 0.5\text{x}10^2 \\ 0.2\text{x}10^1 & -0.75 \end{bmatrix}$$

$$H_2' = \begin{bmatrix} 0.71597737\text{x}10^{-2} & 0.33112583 \\ 0.13245033\text{x}10^{-1} & 0.0 \end{bmatrix} \quad (128)$$

Applying the mixed method and using H_1', H_2', then the reduced model for the modified system $[T_o'(S)]$ is

$$[T_d'(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.985\text{x}10^{-2}S+0.15\text{x}10^{-1} & -0.99999\text{x}10^{-2}S+0.1\text{x}10 \\ -0.4\text{x}10^{-3}S+0.4\text{x}10^{-1} & 0.0 \end{bmatrix} \tag{129}$$

where $\Delta_2(S) = S^2+3S+2$.

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}'(S)] = \frac{1}{\Delta_2(S)} \begin{bmatrix} 0.985\text{x}10^{-2}S+0.15\text{x}10^{-1} \\ -0.4\text{x}10^{-3}S+0.4\text{x}10^{-1} \end{bmatrix} \tag{130}$$

Examining results obtained in Eqs. (126) and (130), the conclusion can be drawn that the values of the constant matrix R can be chosen arbitrarily without affecting the final approximation of the original system.

Applying the unit-step input to Eqs. (122) and (130) yields

$$G_{o1}(S) = \frac{S+1.5}{S^3+103S^2+302S+200} \cdot \frac{1}{S}$$

$$G_{o2}(S) = \frac{4}{S^3+103S^2+302S+200} \cdot \frac{1}{S}$$

$$T_{d11}(S) = \frac{0.00985S+0.015}{S^2+3S+2} \cdot \frac{1}{S}$$

$$T_{d12}(S) = \frac{-0.000S+0.04}{S^2+3S+2} \cdot \frac{1}{S} \tag{131}$$

The unit-step responses are shown in Fig. 16. The steady-state responses are reproduced exactly, while the initial responses of the original model and the reduced

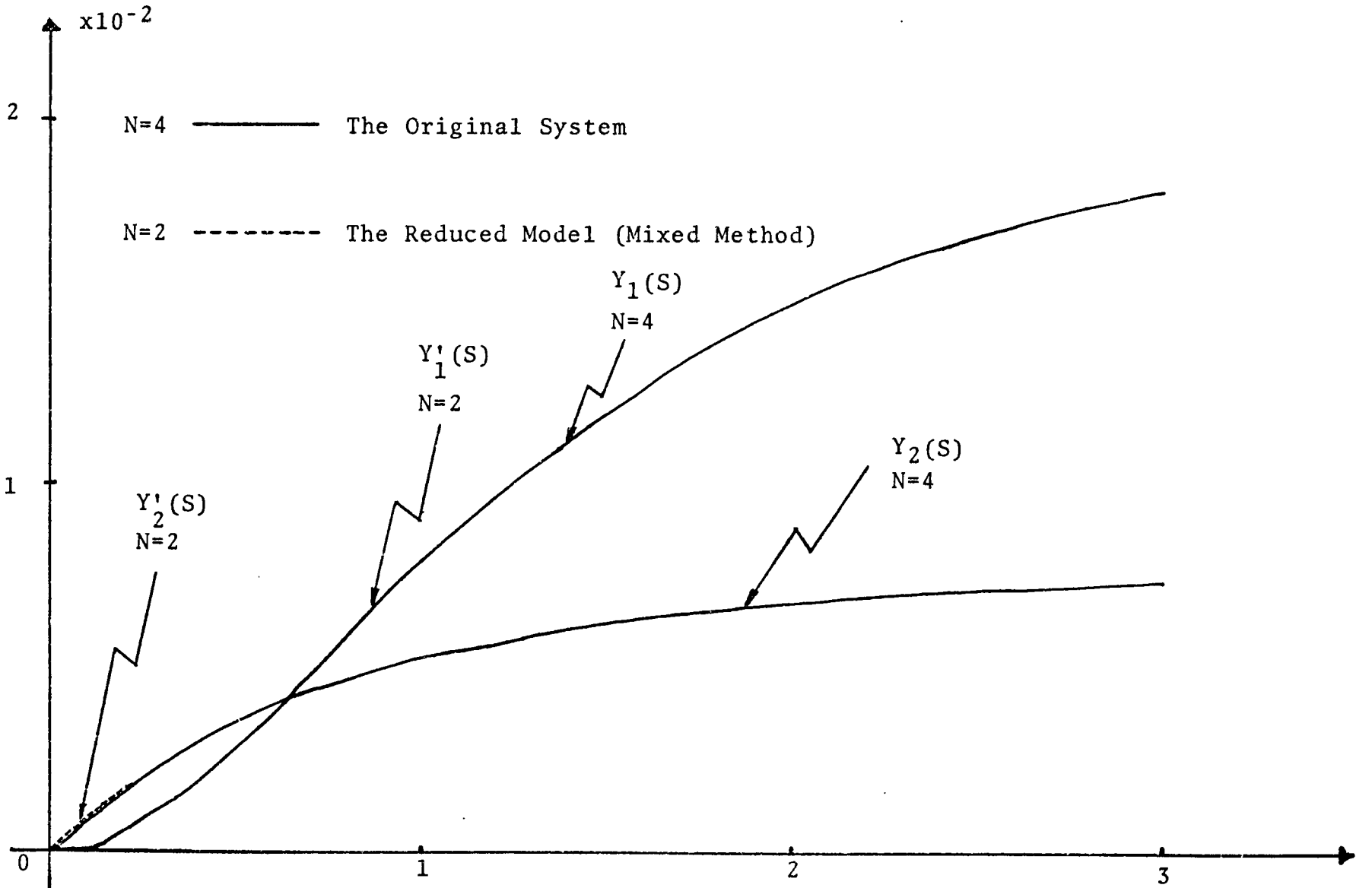

Figure 16. Unit Step Response for Example 10.

model are also very close. Numerically, we can also examine the accuracy of the approximation by comparing the integral square value:

	I.S.V.
First Curve of original system	$0.349244219 \times 10^{-4}$
Second curve of original system	$0.133307446 \times 10^{-3}$
First curve of reduced model	$0.349204055 \times 10^{-4}$
Second curve of reduced model	$0.13335999 \times 10^{-3}$ (131a)

Example 11

To illustrate the application of the above proposed procedures for model simplification of multivariable system when the number of outputs exceeds the number of inputs, consider the following transfer-function matrix $[G_o(S)]$ with $m = 2$ and $\ell = 3$

$$[G_o(S)] = \frac{1}{\Delta_o(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) \\ G_{21}(S) & G_{22}(S) \\ G_{31}(S) & G_{32}(S) \end{bmatrix} \tag{132}$$

where

$$\Delta_o(S) = S^8+218.893S^7+17913.599S^6+634530.95S^5$$
$$+8585434.107S^4+15973802.35S^3+14969803.56S^2$$
$$+7841305.39S+1543146.7$$
$$= (S+0.395)(S+0.405+j0.703)(S+0.405-j0.703)$$
$$(S+0.808)(S+33.32)(S+37.44)(S+73.06+j23.46)$$
$$(S+73.06-j23.46)$$

$$G_{11}(S) = -0.23931\text{x}10^{-1}S^5+0.24787S^4-0.43675\text{x}10^4S^3$$
$$-0.45974\text{x}10^6S^2-0.107847\text{x}10^6S-0.740371\text{x}10^4$$

$$G_{12}(S) = 0.5208\text{x}10^{-2}S^6+0.158711S^5+0.346191\text{x}10^2S^4$$
$$+0.23055\text{x}10^4S^3+0.255857\text{x}10^5S^2-0.148485\text{x}10^6S$$
$$-0.165375\text{x}10^4$$

$$G_{21}(S) = S^7-0.23931\text{x}10^{-1}S^6+0.24787\text{x}10S^5-0.436752\text{x}10^3S^4$$
$$-0.45974\text{x}10^5S^3-0.107848\text{x}10^7S^2-0.74037\text{x}10^6S$$
$$+0.14226\text{x}10^2$$

$$G_{22}(S) = 0.5208\text{x}10^{-3}S^7+ 0.15871\text{x}10^{-1}S^6+ 0.34619\text{x}10^2S^5$$
$$+0.23055\text{x}10^3S^4+0.25586\text{x}10^4S^3-0.14848\text{x}10^5S^2$$
$$-0.16537\text{x}10^6S-0.14265\text{x}10^2$$

$$G_{31}(S) = 0.2699\text{x}10^{-4}S^7+0.49139\text{x}10^{-2}S^6+0.14447\text{x}10S^5$$
$$+0.10672\text{x}10^3S^4+0.22932\text{x}10^4S^3+0.148882\text{x}10^7S^2$$
$$+0.23291\text{x}10^6S+0.289138\text{x}10^3$$

$$G_{32}(S) = 0.1297\text{x}10^{-3}S^7+0.12278\text{x}10^{-1}S^6+0.264899\text{x}10^2S^5$$
$$+0.259459\text{x}10^3S^4+0.637956\text{x}10^5S^3+0.37376\text{x}10^6S^2$$
$$+0.67494\text{x}10^6S+0.360133\text{x}10^4$$

The characteristic polynomial of the matrix $[G_o(S)]$ is $\Delta(S) = \{\Delta_o(S)\}^2$. The following procedures are performed to

obtain a new modified transfer-function matrix:

$$[T_o(S)] = G_1(S)+[G_2(S)]$$

$$= \frac{1}{\Delta_o(S)} \left[\begin{array}{cc|c} G_{11}(S) & G_{12}(S) & 0 \\ G_{21}(S) & G_{22}(S) & 0 \\ G_{31}(S) & G_{32}(S) & 1 \end{array}\right] \tag{133}$$

The rank of $[T_o(S)]$ is 24. The ratio of the rank and the dimension of this modified system is $\frac{24}{3} = 8 = K$, an integer. By using Eq. (37), we have $2K = 16$ matrix quotients. The first four matrix quotients are:

$$H_1 = \begin{bmatrix} 0.0 & 0.10039151\text{x}10^{6} & 0.39765446\text{x}10^{3} \\ 0.0 & -0.80600783\text{x}10^{4} & 0.39656729\text{x}10^{3} \\ 0.15431467\text{x}10^{7} & 0.7299403\text{x}10^{9} & 0.35999415\text{x}10^{3} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} -0.93675833\text{x}10^{-3} & -0.11630684\text{x}10^{-3} & 0.12752979\text{x}10^{-6} \\ 0.566187\text{x}10^{-8} & 0.6761924\text{x}10^{-7} & 0.67457074\text{x}10^{-28} \\ -0.11086398\text{x}10^{-5} & -0.13630286\text{x}10^{-4} & -0.10907614\text{x}10^{-25} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0.18925893\text{x}10^{-16} & -0.10040056\text{x}10^{6} & -0.39963943\text{x}10^{3} \\ -0.3713575\text{x}10^{-16} & 0.80677805\text{x}10^{4} & -0.38428994\text{x}10^{3} \\ -0.41073398\text{x}10^{7} & -0.72966072\text{x}10^{9} & -0.39622133\text{x}10^{7} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} -0.28310922\text{x}10^{-1} & -0.41689233\text{x}10^{-2} & -0.28914324\text{x}10^{-6} \\ -0.10361655 & -0.96124859\text{x}10^{-2} & -0.34176715\text{x}10^{-21} \\ 0.10004233 & 0.58568629\text{x}10^{-1} & 0.22997037\text{x}10^{-21} \end{bmatrix} \tag{134}$$

The mixed method is applied for the approximation and H_i, $i = 1,\ldots 4$ are used; then the reduced model for the modified system $[T_o(S)]$ is

$$[T_d(S)] = \frac{1}{\Delta_4(S)} \left[\begin{array}{cc|c} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \\ T_{31}(S) & T_{32}(S) & T_{33}(S) \end{array}\right] \tag{135}$$

where $\Delta_4(S) = (S+0.395)(S+0.405+j0.703)(S+0.405-j0.703)$
$(S+0.808)$

$= S^4+2.013S^3+1.952S^2+1.0504S+0.21008$

$T_{11}(S) = 0.44851083\text{x}10^{-2}S^3-0.61364882\text{x}10^{-1}S^2$
$-0.14596779\text{x}10^{-1}S-0.1007922\text{x}10^{-2}$

$T_{12}(S) = -0.55124746\text{x}10^{-4}S^3+0.51334301\text{x}10^{-2}S^2$
$-0.20195333\text{x}10^{-1}S-0.22513723\text{x}10^{-3}$

$T_{13}(S) = -0.29165322\text{x}10^{-9}S^3+0.59350573\text{x}10^{-9}S^2$
$-0.110780014\text{x}10^{-7}S+0.13613741\text{x}10^{-6}$

$T_{21}(S) = 0.45855403\text{x}10^{-2}S^3-0.13832512S^2-0.10079229S$
$+0.19366969\text{x}10^{-5}$

$T_{22}(S) = 0.17702483\text{x}10^{-3}S^3-0.1237636\text{x}10^{-3}S^2$
$-0.22512904\text{x}10^{-1}S-0.19420003\text{x}10^{-5}$

$T_{23}(S) = 0.57635563\text{x}10^{-20}S^3-0.25594053\text{x}10^{-20}S^2$
$-0.80537956\text{x}10^{-22}S+0.0$

$T_{31}(S) = -0.16290698\text{x}10^{-1}S^3+0.20001136S^2$
$+0.3170455\text{x}10^{-1}S+0.393625\text{x}10^{-4}$

$T_{32}(S) = 0.49174273\text{x}10^{-2}S^3+0.43387104\text{x}10^{-1}S^2$
$+0.9184469\text{x}10^{-1}S+0.49027575\text{x}10^{-3}$

$T_{33}(S) = -0.10947895\text{x}10^{-20}S^3+0.14815543\text{x}10^{-20}S^2$
$+0.79338464\text{x}10^{-22}S+0.34754821\text{x}10^{-24}$

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{1}{\Delta_4(S)} \begin{bmatrix} T_{11}(S) & T_{12}(S) \\ T_{21}(S) & T_{22}(S) \\ T_{31}(S) & T_{32}(S) \end{bmatrix} \tag{136}$$

The characteristic polynomial of $[T_{d1}(S)]$ is $\Delta(S) = \{\Delta_4(S)\}^2$. Expand the original matrix equation into scaler input-output expressions and the unit-step responses are as follows:

$$G_{o1}(S) = \frac{\begin{bmatrix} 0.5208\text{x}10^{-2}S^6+0.13478S^5+0.3486697\text{x}10^{-2}S^4 \\ -0.2062\text{x}10^4S^3-0.4341543\text{x}10^6S^2 \\ -0.2563325S-0.905746 \end{bmatrix}}{\Delta_o(S)} \cdot \frac{1}{S}$$

$$G_{o2}(S) = \frac{\begin{bmatrix} 0.5208\text{x}10^{-3}S^7-0.0806\text{x}10^{-1}S^6 \\ +0.370977\text{x}10^2S^5-0.206202\text{x}10^3S^4 \\ -0.434154\text{x}10^5S^3-0.1093328\text{x}10^7S^2 \\ -0.90574\text{x}10^6S-0.00039\text{x}10^2 \end{bmatrix}}{\Delta_o(S)} : \frac{1}{S}$$

$$G_{o3}(S) = \frac{\begin{bmatrix} 0.15669\text{x}10^{-3}S^7+0.171919\text{x}10^{-1}S^6 \\ +0.279346\text{x}10^2S^5+0.366179\text{x}10^3S^4 \\ +0.66088\text{x}10^5S^3+0.186258\text{x}10^7S^2 \\ +0.90785\text{x}10^6S+0.3890468\text{x}10^4 \end{bmatrix}}{\Delta_o(S)} \cdot \frac{1}{S}$$

The unit-step responses of the reduced model are

$$T_{d11}(S) = \frac{0.4429984\times10^{-2}S^3-0.5623145\times10^{-1}S^2 -0.347921\times10^{-1}S-0.1233059\times10^{-2}}{\Delta_4(S)} \cdot \frac{1}{S}$$

$$T_{d12}(S) = \frac{0.4762564\times10^{-2}S^3-0.1384488S^2 -0.1233052S-0.5309\times10^{-8}}{\Delta_4(S)} \cdot \frac{1}{S}$$

$$T_{d13}(S) = \frac{-0.1137327\times10^{-1}S +0.2433984S +0.12354924S+0.5296382\times10^{-3}}{\Delta_4(S)} \cdot \frac{1}{S} \quad (137)$$

The comparisons of the responses of the original system and reduced model are shown graphically in Figs. 17, 18, and 19. Again, the accuracy of approximation by mixed method may be examined numerically. The corresponding integral square values are:

	I.S.V.
First curve of original system	0.13386961×10^{-2}
Second curve of original system	0.12036692×10^{-1}
Third curve of original system	0.21931316×10^{-1}
First curve of reduced model	0.1347451×10^{-2}
Second curve of reduced model	0.11968612×10^{-1}

Third curve of reduced

model $0.22000029 \times 10^{-1}$ (138)

The model simplification by the method of matrix-continued fraction is far from satisfactory for this particular system. A study of the previous examples reveals that when the poles of the original system are located close to each other in the S-plane, the second Cauer form approximation yields better results. However, if some poles are far from the jw-axis of the S-plane while others are close to the jw-axis, the mixed method gives a better approximation than the second Cauer form. The success of these approaches depends upon the numbers of dominant-matrix quotients used.

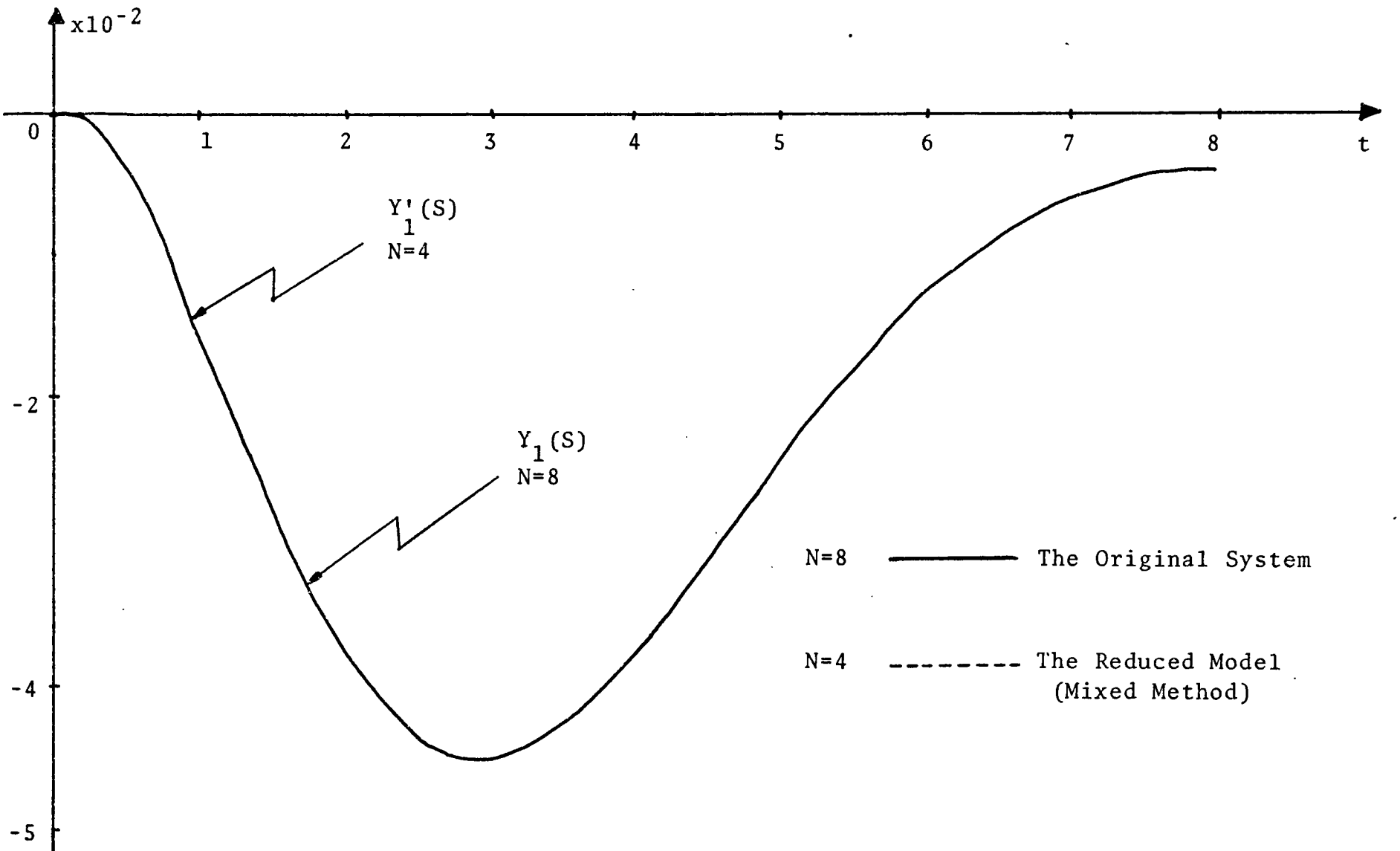

Figure 17. Unit Step Response for Example 11, First Curve.

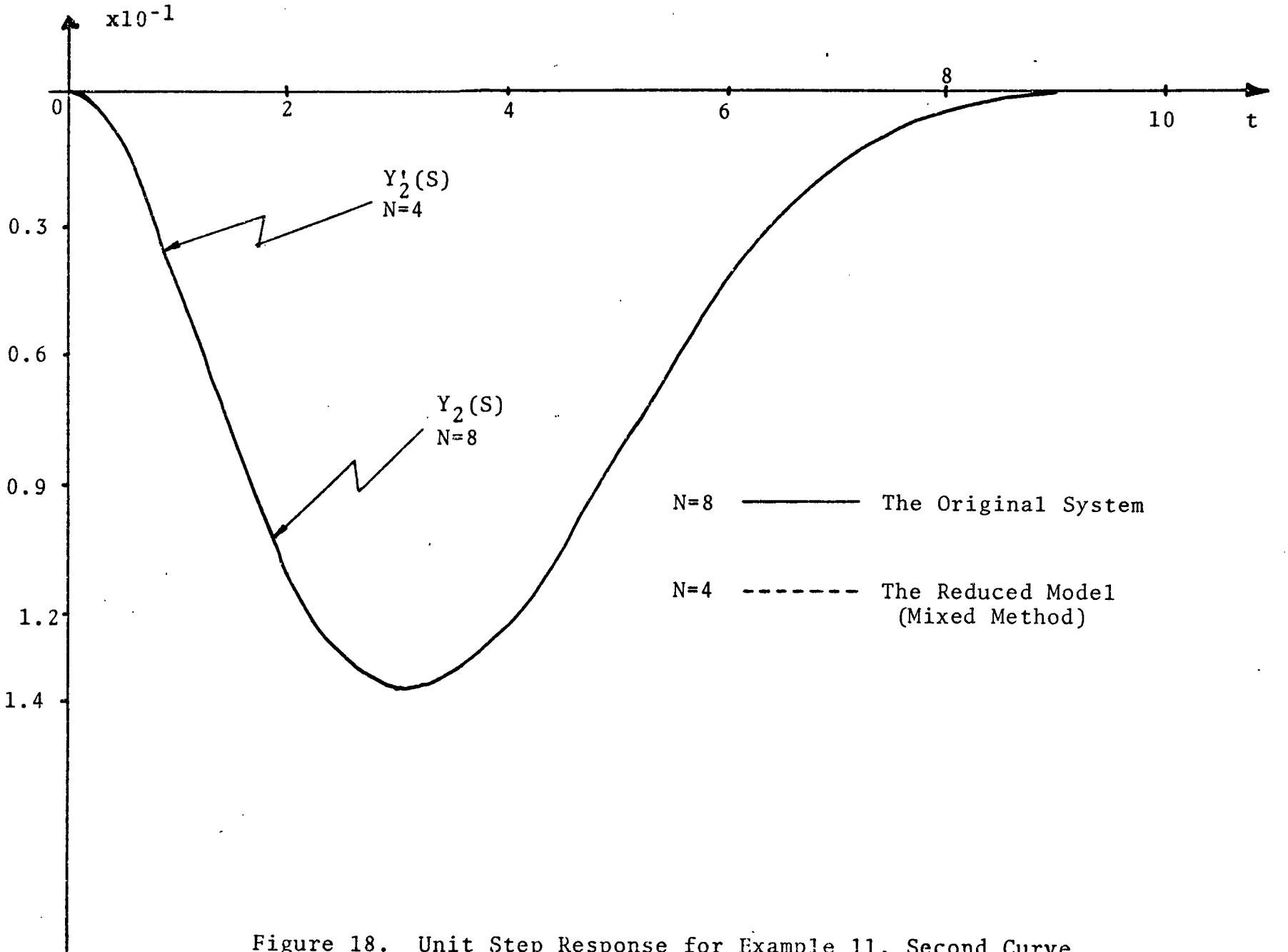

Figure 18. Unit Step Response for Example 11, Second Curve.

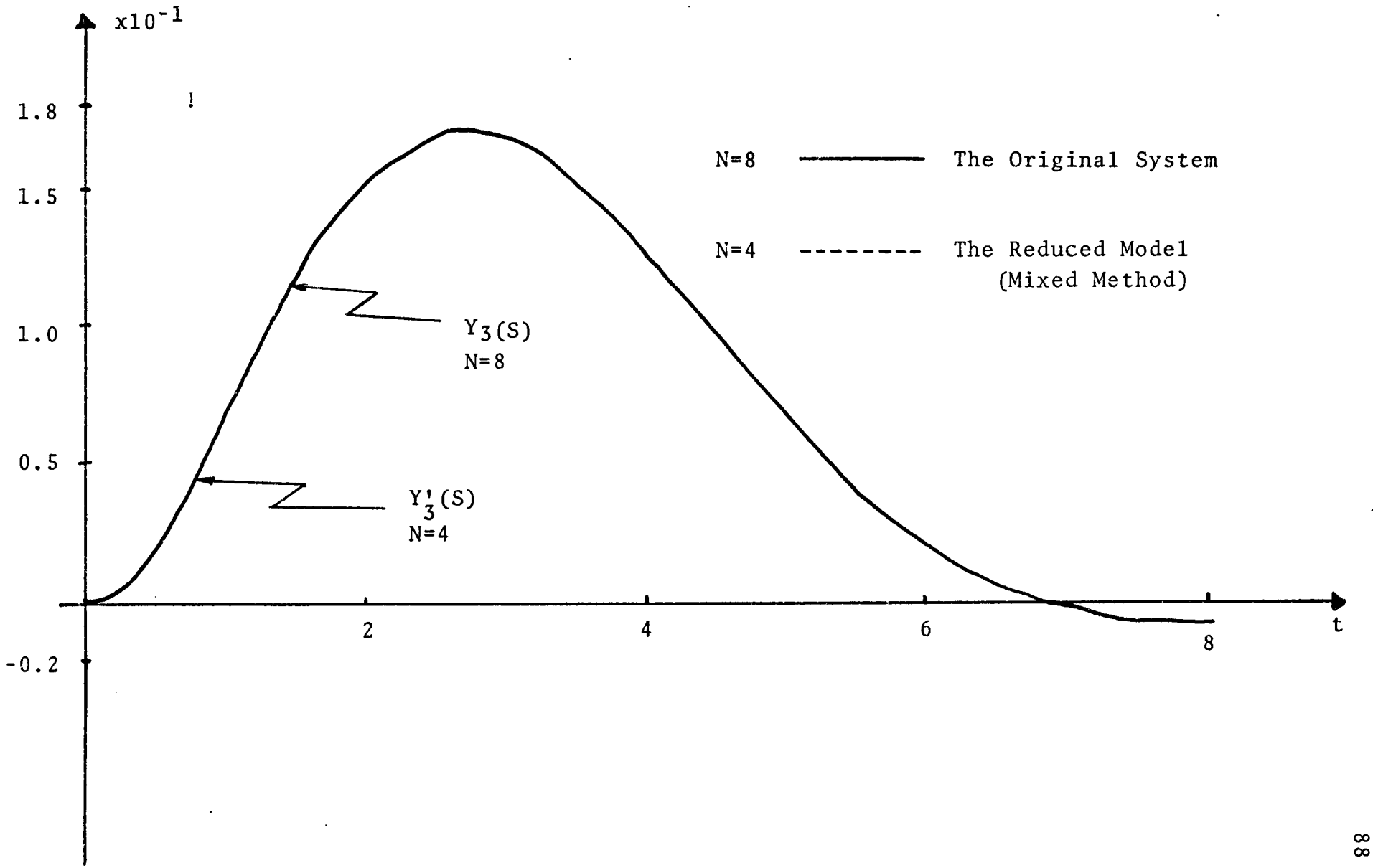

Figure 19. Unit Step Response for Example 11, Third Curve.

CHAPTER IV

ILL-CONDITIONED CASES

In the previous chapters, it has been shown that the realization of the matrix $[G_0(S)]$ is a completely controllable and completely observable system with minimal dimension Kxm if det $[H_i] \neq 0$, $i = 1, 2, \ldots, 2K$ and $K \leq n$. This means that the rank of $[G_0(S)]$ is Kxm and the degree of the characteristic polynomial of $[G_0(S)]$ (i.e., the least common denominator of all the minors of $[G_0(S)]$) is Kxm. Whenever the rank of $[G_0(S)]$ is not equal to Kxm, the complete set of matrix quotients in Eq. (39) cannot be obtained. On the other hand, if the rank $[G_0(S)]$ = Kxm but det [Ap. 1] = 0, which may occur due to ill-conditioned constants in [Ap. 1], then the matrix Routh algorithm of Eq. (48) cannot be applied.

A method is therefore proposed in this chapter to overcome these ill-conditioned cases. When a system is ill-conditioned, a new transfer-function matrix $[T_0(S)]$ shall be constructed by modifying the $[G_0(S)]$ and by adding another square transfer-function matrix $[G_2(S)]$. This can be expressed by

$$[T_o(S)] = [G_1(S)]+[G_2(S)], \tag{139}$$

where

$$[G_1(S)] = \left[\begin{array}{c|c} \begin{matrix}(m\text{x}m)\\ G_o(S)\end{matrix} & \begin{matrix}(m\text{x}1)\\ 0\end{matrix} \\ \hline \begin{matrix}(1\text{x}m)\\ 0\end{matrix} & \begin{matrix}(1\text{x}1)\\ 0\end{matrix} \end{array}\right]$$

$$= \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} \begin{matrix}(m\text{x}m)\\ Q(S)\end{matrix} & \begin{matrix}(m\text{x}1)\\ 0\end{matrix} \\ \hline \begin{matrix}(1\text{x}m)\\ 0\end{matrix} & \begin{matrix}(1\text{x}1)\\ 0\end{matrix} \end{array}\right]$$

$$[G_2(S)] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} \begin{matrix}(m\text{x}m)\\ 0\end{matrix} & \begin{matrix}(m\text{x}1)\\ R_{12}\end{matrix} \\ \hline \begin{matrix}(1\text{x}m)\\ R_{21}\end{matrix} & \begin{matrix}(1\text{x}1)\\ R_{22}\end{matrix} \end{array}\right]$$

and

$$[T_o(S)] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} \begin{matrix}(m\text{x}m)\\ Q(S)\end{matrix} & \begin{matrix}(m\text{x}1)\\ R_{12}\end{matrix} \\ \hline \begin{matrix}(1\text{x}m)\\ R_{21}\end{matrix} & \begin{matrix}(1\text{x}1)\\ R_{22}\end{matrix} \end{array}\right]$$

[Rij] are constant matrices and the elements in [Rij] should be chosen such that the rank of $[T_o(S)] = K\text{x}(m+1)$, where K is an integer. Applying the matrix Routh algorithm in Eq. (48) yields 2K matrix quotients (H_i, $i = 1,\ldots,2K$). Either the method of matrix-continued fraction or the mixed method can be applied to obtain the reduced model for the modified system $[T_o(S)]$. The reduced model is

$$[Y_d(S)] = [T_d(S)][U_d(S)] \tag{140}$$

where

$$[T_d(S)] = \left[\begin{array}{c|c} \overset{(m\times m)}{T_{11}(S)} & \overset{(m\times 1)}{T_{12}(S)} \\ \hline \overset{(1\times m)}{T_{21}(S)} & \overset{(1\times 1)}{T_{22}(S)} \end{array}\right]$$

$$[\ _d(S)] = \left[\begin{array}{c} \overset{(m\times 1)}{Y_{11}(S)} \\ \hline \overset{(1\times 1)}{Y_{21}(S)} \end{array}\right] + \left[\begin{array}{c} \overset{(m\times 1)}{Y_{21}(S)} \\ \hline \overset{(1\times 1)}{Y_{22}(S)} \end{array}\right]$$

and

$$[U_d(S)] = \left[\begin{array}{c} \overset{(m\times 1)}{U_o(S)} \\ \hline \overset{(1\times 1)}{U_1(S)} \end{array}\right]$$

The reduced model of the original system $[G_o(S)]$ can be obtained by partitioning the matrix in Eq. (140), or

$$\begin{aligned} [Y_{11}(S)] &= [T_{11}(S)][U_o(S)] \\ &\simeq [Y_o(S)] = [G_o(S)][U_o(S)] \end{aligned} \tag{141}$$

<u>Example 12</u>

To apply the matrix Routh algorithm in Eq. (48) effectively, consider the following simple multivariable system:

$$\begin{aligned} [Y_o(S)] &= [G_o(S)][U_o(S)] \\ &= \frac{1}{\Delta_o(S)}\,[Q_1][U_o(S)] \end{aligned} \tag{142}$$

where $\Delta_o(S) = (S+1)(S+2)$

and

$$[Q_1] = \begin{bmatrix} 1. & 1. \\ 1. & 1. \end{bmatrix}$$

The rank of $[G_o(S)] = r_o = 2$ and the dimension of $[G_o(S)]$ is $1 = m = 2$. Since the ratio of r_o and m is $K = \frac{r_o}{m} = 1$, an integer, it is expected that the matrix Routh algorithm can be applied to obtain $2K = 2$ matrix quotients. However, the system has singular steady-state gain, or $\det [A_{21}] = \det [Q_1] = 0$. This is an ill-conditioned case. The proposed method stated in Eq. (139) can be applied to evaluate the matrix quotients.

The new transfer-function matrix $[T_o(S)]$ in Eq. (139) can be constructed as:

$$[T_o(S)] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} Q & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right] \tag{143}$$

$$= \frac{1}{(S+1)(S+2)} \left[\begin{array}{cc|c} 1. & 1. & 1. \\ 1. & 1. & 0 \\ \hline 1. & 0 & 1. \end{array}\right]$$

The rank of $[T_o(S)]$ is $6 = K\mathrm{x}(m+1)$, where $m = 2$. The ratio $\frac{6}{m+1} = K$ is 2. Therefore, we have $2K = 4$ matrix quotients as follows:

$$H_1 = 2 \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \qquad H_2 = 1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$H_3 = -9 \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{and} \qquad H_4 = -1/3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{144}$$

To check the result, we substitute Eq. (144) into Eq. (39) and partition the resulting matrix in Eq. (140). The original system matrix can be obtained from Eq. (141), or

$$[T_o(S)] = [(H_2+H_4)S+H_2H_3H_4][S^2I+(H_1H_2+H_1H_4+H_3H_4)S+H_1H_2H_3H_4]^{-1}$$

$$= \left[\begin{array}{c|c} T_{11}(S) & T_{12}(S) \\ \hline T_{21}(S) & T_{22}(S) \end{array}\right] \tag{145}$$

The original system matrix is

$$[G_o(S)] = [T_{11}(S)]$$

$$= \frac{1}{(S+1)(S+2)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{146}$$

It can be seen that Eq. (142) and Eq. (146) eventually become the same.

A demonstration of the techniques established in this chapter and an illustration of the power of this approach in approximating a multivariable system with an ill-conditioned numerical element is given by the following example.

<u>Example 13</u>

We desire to approximate a two-inputs and two-outputs multivariable system

$$[Y_o(S)] = [G_o(S)][U_o(S)] \tag{147}$$

The transfer-function matrix

$$[G_o(S)] = \frac{1}{\Delta_o(S)}[Q_1(S)] = \frac{1}{\Delta_o(S)} \begin{bmatrix} G_{11}(S) & G_{12}(S) \\ G_{21}(S) & G_{22}(S) \end{bmatrix} \tag{148}$$

where

$$\Delta_o(S) = S^8+30.41S^7+358.4295S^6+2913.8638S^5$$
$$+18110.567S^4+67556.983S^3+173383.58S^2$$
$$+149172.19S+37752.826$$
$$= (S+0.35+j6.8)(S+0.35-j6.8)(S+0.46)$$
$$(S+0.75)(S+2.2+j3.6)(S+2.2-j3.6)(S+8.5)$$
$$(S+15.6)$$

$$G_{11}(S) = 31.6S^7+755.1355S^6+9277.6041S^5+76590.871S^4$$
$$+409701.2S^3+1564162.3S^2+3384185.25S$$
$$+1413173.6$$

$$G_{12}(S) = 18.4S^7+483.6325S^6+7473.1599S^5+46751.11S^4$$
$$+257757.71S^3+975359.78S^2+2226559.5S$$
$$+1413173.6$$

$$G_{21}(S) = 5.16S^7+115.8905S^6+1530.8447S^5+12384.435S^4$$
$$+66377.492S^3+229694.04S^2+384459.57S$$
$$+321603.99$$

$$G_{22}(S) = 18.0S^7+448.3985S^6+4574.0688S^5+36124.091S^4$$
$$+173628.46S^3+594325.26S^2+1185233.9S$$
$$+321603.99$$

Note that det $[A_{2,1}] = 0$. This is an ill-conditioned case and the matrix Routh algorithm of Eq. (37) cannot be applied. By using the proposed method in this chapter, the new transfer-function matrix $[T_o(S)]$ can be constructed:

$$[T_o(S)] = \frac{1}{\Delta_o(S)} \left[\begin{array}{c|c} Q_1(S) & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right]$$

$$= \frac{1}{\Delta_o(S)} \left[\begin{array}{cc|c} Q_{11}(S) & Q_{12}(S) & 1. \\ Q_{21}(S) & Q_{22}(S) & 0. \\ \hline 1. & 0. & 1. \end{array}\right] \tag{149}$$

The rank of $[T_o(S)]$ is $24 = K(m+1)$, where $m = 2$, and the ratio $\frac{24}{m+1} = K$ is 8. Hence, the matrix Routh algorithm can be applied to evaluate the $2K = 16$ matrix quotients. The first four matrix quotients are:

$$H_1 = \begin{bmatrix} -0.37752826\text{x}10^5 & 0.16589128\text{x}10^6 & 0.37752826\text{x}10^5 \\ 0.37752826\text{x}10^5 & -0.16589116\text{x}10^6 & -0.37752826\text{x}10^5 \\ 0.37752826\text{x}10^5 & -0.16589128\text{x}10^6 & 0.0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0.20196217\text{x}10^2 & 0.20196221\text{x}10^2 & 0.67036624\text{x}10^{-5} \\ 0.4596169\text{x}10 & 0.45961715\text{x}10 & 0.0 \\ 0.67036624\text{x}10^{-5} & 0.25630793\text{x}10^{-16} & 0.67036624\text{x}10^{-5} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0.37753022\text{x}10^5 & -0.16589115\text{x}10^6 & -0.37753022\text{x}10^5 \\ -0.37752769\text{x}10^5 & 0.16589141\text{x}10^6 & 0.37752769\text{x}10^5 \\ -0.37753022\text{x}10^5 & 0.16589115\text{x}10^6 & -0.90588671\text{x}10^5 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 0.6994176\text{x}10 & -0.44283034\text{x}10 & -0.10084175\text{x}10^{-4} \\ -0.10486528\text{x}10 & 0.62444541\text{x}10 & -0.57922887\text{x}10^{-16} \\ -0.10084175\text{x}10^{-4} & 0.45504833\text{x}10^{-15} & -0.10084175\text{x}10^{-4} \end{bmatrix} \tag{150}$$

If the method of the matrix-continued fraction is applied and H_1, and H_2 are used, then the reduced model for this new transfer-function matrix is

$$[T_d(S)] = \frac{1}{\Delta_3(S)} \left[\begin{array}{cc|c} T_{11}(S) & T_{12}(S) & T_{13}(S) \\ T_{21}(S) & T_{22}(S) & T_{23}(S) \\ \hline T_{31}(S) & T_{32}(S) & T_{33}(S) \end{array}\right] \tag{151}$$

where $\Delta_3(S)$ = The characteristic polynomial of the reduced model

$$= S^3+0.7926229S^2+0.13654829S+0.29197843\times10^{-7}$$

$$= (S+0.53954056)(S+0.0000002138)(S+0.2530822157)$$

$$T_{11}(S) = 0.20196217\times10^2S^2+0.51113092\times10S+0.10929473\times10^{-5}$$

$$T_{12}(S) = 0.20196221\times10^2S^2+0.51113083\times10S+0.10929306\times10^{-5}$$

$$T_{13}(S) = 0.67036624\times10^{-5}S^2+0.36168993\times10^{-5}S+0.77339491\times10^{-12}$$

$$T_{21}(S) = 0.4596169\times10S^2+0.11632095\times10S+0.24872827\times10^{-6}$$

$$T_{22}(S) = 0.45961715\times10S^2+0.11632102\times10S+0.24872446\times10^{-6}$$

$$T_{23}(S) = 0.0$$

$$T_{31}(S) = 0.67036624\times10^{-5}S^2+0.36168993\times10^{-5}S+0.77339491\times10^{-12}$$

$$T_{32}(S) = 0.25630793\times10^{-16}S^2+0.21175824\times10^{-21}S+0.18528846\times10^{-21}$$

$$T_{33}(S) = 0.67036624\times10^{-5}S^2+0.36168993\times10^{-5}S+0.77339491\times10^{-12}$$

The reduced model for the original system $[G_o(S)]$ is

$$T_{d1}(S) = \frac{1}{\Delta_3(S)}\begin{bmatrix} T_{11}(S) & T_{12}(S) \\ T_{21}(S) & T_{22}(S) \end{bmatrix} \qquad (152)$$

If the mixed method is applied and fourth-order reduced model is required, then H_i, $i = 1,\ldots,4$ are used. The reduced model for the new modified transfer-function matrix is:

$$[T_d(S)] = \frac{1}{\Delta_4(S)} \begin{bmatrix} T_{11}^*(S) & T_{12}^*(S) & T_{13}^*(S) \\ T_{21}^*(S) & T_{22}^*(S) & T_{23}^*(S) \\ T_{31}^*(S) & T_{32}^*(S) & T_{33}^*(S) \end{bmatrix} \tag{153}$$

where $\Delta_4(S)$ = The least common-denominator polynomial

$$= (S+0.35+j6.8)(S+0.35-j6.8)(S+0.46)(S+0.75)$$

$$= S^4+1.91S^3+47.5545S^2+56.340125S+15.9950625$$

$$T_{11}^*(S) = 0.2272287\text{x}10^2S^3+0.94651495\text{x}10^2S^2 +0.10011058\text{x}10^4S+0.58570481\text{x}10^3$$

$$T_{12}^*(S) = 0.22753554\text{x}10^2S^3+0.9468806\text{x}10^2S^2 +0.10011174\text{x}10^4S+0.58570481\text{x}10^3$$

$$T_{13}^*(S) = 0.93723182\text{x}10^{-6}S^3+0.31920686\text{x}10^{-4}S^2 -0.18173293\text{x}10^{-3}S+0.42367855\text{x}10^{-3}$$

$$T_{21}^*(S) = 0.51793219\text{x}10S^3+0.21551644\text{x}10^2S^2 +0.22783097\text{x}10^3S+0.13329219\text{x}10^3$$

$$T_{22}^*(S) = 0.51593345\text{x}10S^3+0.21527749\text{x}10^2S^2 +0.22782415\text{x}10^3S+0.13329219\text{x}10^3$$

$$T_{23}^*(S) = -0.97699626\text{x}10^{-12}S^3+0.47369516\text{x}10^{-12}S^2 -0.47369516\text{x}10^{-12}S-0.33870862\text{x}10^{-20}$$

$$T_{31}^*(S) = 0.9372305\text{x}10^{-6}S^3+0.31920686\text{x}10^{-4}S^2 -0.18173293\text{x}10^{-3}S+0.42367855\text{x}10^{-3}$$

$$T_{32}^{*}(S) = -0.68454686\times10^{-11}S^3+0.28244629\times10^{-11}S^2$$
$$-0.15788126\times10^{-11}S+0.0$$
$$T_{33}^{*}(S) = 0.93723611\times10^{-6}S^3+0.31920683\times10^{-4}S^2$$
$$-0.18173293\times10^{-3}S+0.42367855\times10^{-3}$$

The reduced model for the original system $[G_o(S)]$ is

$$[T_{d1}(S)] = \frac{1}{\Delta_4(S)}\begin{bmatrix} T_{11}^{*}(S) & T_{12}^{*}(S) \\ T_{21}^{*}(S) & T_{22}^{*}(S) \end{bmatrix} \tag{154}$$

The comparison of the unit-step response curves of the original system and the reduced models in Eqs. (148), (152), and (154) are shown in Figs. 20 and 21. The reduced models are stable and have a good approximation in the steady-state responses.

The integral square values of the original system, second Cauer form approximation, and mixed method approximation are obtained by applying Katz's formula:

	I.S.V.
First curve of original system	0.16639655×10^4
Second curve of original system	0.125640808×10^3
First curve of second Caver form reduced model	0.18585996×10^4

Second curve of second Caver form reduced model	$0.962381744 \times 10^{2}$
First curve of mixed method reduced model	$0.151197949219 \times 10^{4}$
Second curve of mixed method reduced model	0.78306518×10^{2} (155)

From Eq. (155), we see that the approximation, as a whole, is satisfactory.

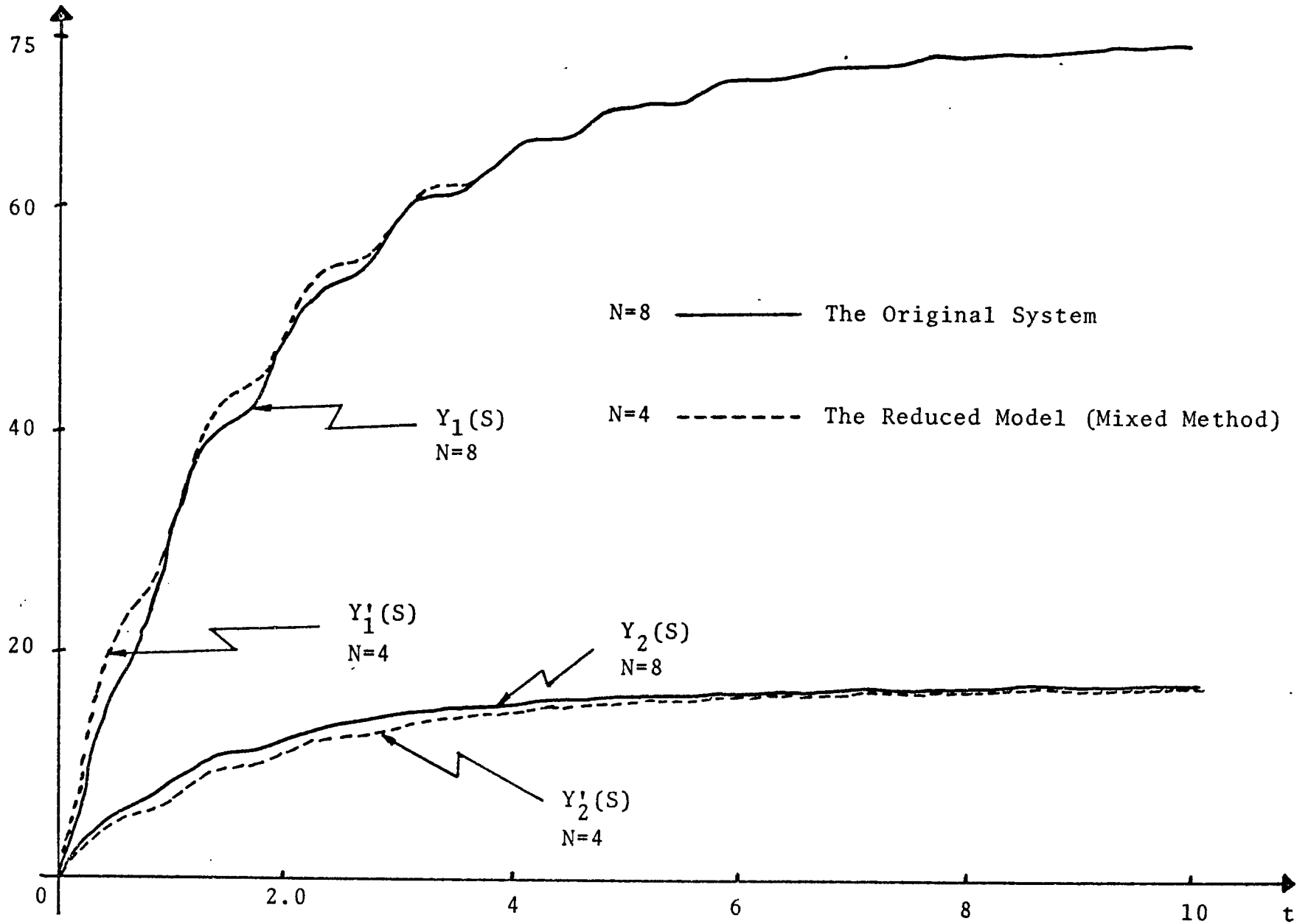

Figure 20. Unit Step Response for Example 13, First Curve.

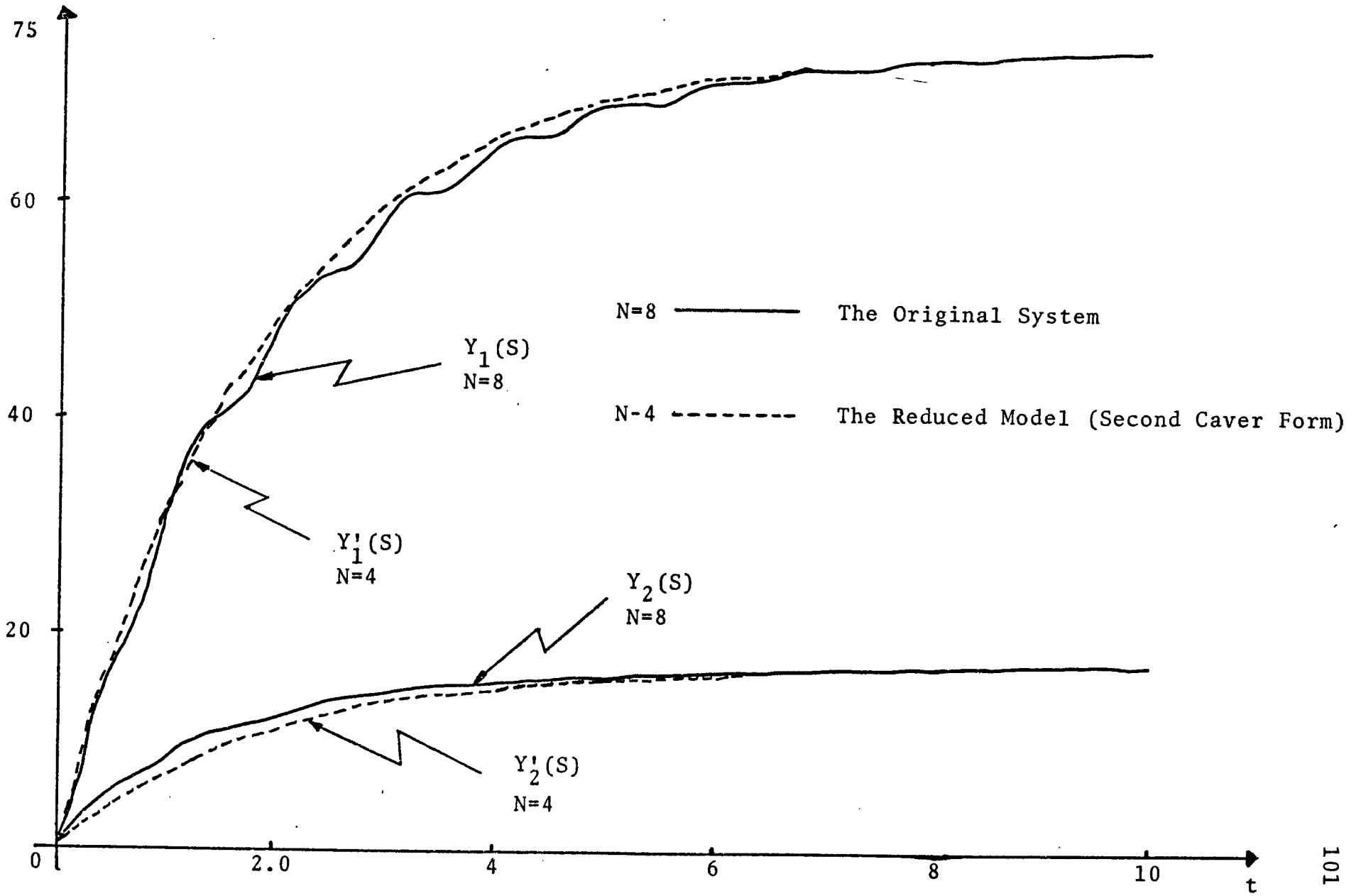

Figure 21. Unit Step Response for Example 13, Second Curve.

CHAPTER V

CONCLUSION

An algebraic method has been proposed in the frequency domain to obtain a stable reduced model of a high-degree multivariable system with m inputs and ℓ outputs, where m is not necessarily equal to ℓ. The proposed mixed method uses the advantages of both the matrix-continued fraction approach and the dominant-eigenvalue concept. Use of the matrix-continued fraction method for the simplification of multivariable systems having an equal number of inputs and outputs has been extended to simplify high-degree multivariable systems with various numbers of inputs and outputs. The methods are simple in theory and flexible in practice. The reduced models provide a good approximation if all inputs are excited by the same signals. The success of these approaches depends on the numbers of dominant-matrix quotients used; therefore, the proposed approaches are particularly suitable for the reduction of high-degree multivariable systems with small numbers of inputs and outputs. The whole process can be performed by a digital computer.

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```
C
C       THIS PROGRAM FINDS THE ROOTS OF A POLYNOMIAL WITH
C       REAL COEFFICIENTS.
C       M ... DEGREE OF THE POLYNOMIAL.
C       ROOTR ... REAL ROOT ARRAY.
C       ROOTI ... IMAGINARY ROOT ARRAY.
C       THE COEFFICIENTS ARE READ IN DESCENDING ORDER.
C

        DOUBLE PRECISION COEF(10),WORK(10),ROOTR(10),ROOTI(10)
        READ(5,33)M
        N=M+1
        READ(5,34)(COEF(I),I=1,N)
   33   FORMAT(I2)
   34   FORMAT((4F20.8))
        DO 5 J=1,N
        WRITE(6,100) COEF(J)
  100   FORMAT(F20.10)
    5   CONTINUE
        CALL PROOT(COEF,WORK,M,ROOTR,ROOTI,IER)
        IF(IER.NE.0) GO TO 10
        DO 20 I=1,M
        WRITE(6,50) ROOTR(I),ROOTI(I)
   50   FORMAT(//F20.10,10X,F20.10)
   20   CONTINUE
        GO TO 30
   10   WRITE(6,40)
   40   FORMAT(2X,'THERE IS NO SOLUTION')
   30   STOP
        END



        SUBROUTINE PROOT(COEF,WORK,M,ROOTR,ROOTI,IER)
        DOUBLE PRECISION COEF(1),WORK(1),ROOTR(1),ROOTI(1)
        IFIT=0
        N=M
        IER=0
        NX=N
        NXX=N+1
        N2=1
        KJ1=N+1
        DO 40 L=1,KJ1
        MT=KJ1-L+1
40      WORK(MT)=COEF(L)
45      X0=.00500101
        Y0=.01000101
        IN=0
50      X=X0
        X0=-10.0*Y0
        Y0=-10.0*X
        X=X0
        Y=Y0
        IN=IN+1
        GO TO 59
55      IFIT=1
        XPR=X
        YPR=Y
59      ICT=0
60      UX=0.0
        UY=0.0
        V=0.0
        YT=0.0
        XT=1.0
        U=WORK(N+1)
```

```
      IF(U) 65,130,65
65    DO 70 I=1,N
      L=N-I+1
      XT2=X*XT-Y*YT
      YT2=X*YT+Y*XT
      U=U+WORK(L)*XT2
      V=V+WORK(L)*YT2
      UX=UX+I*XT*WORK(L)
      UY=UY-I*YT*WORK(L)
      XT=XT2
70    YT=YT2
      SUMSQ=UX*UX+UY*UY
      IF (SUMSQ) 75,110,75
75    DX=(V*UY-U*UX)/SUMSQ
      X=X+DX
      DY=-(U*UY+V*UX)/SUMSQ
      Y=Y+DY
78      IF(ABS(DY)+ABS(DX)-1.0E-05) 100,80,80
80    ICT=ICT+1
      IF (ICT-500) 60,85,85
85     IF(IFIT) 100,90,100
90    IF(IN-5) 50,95,95
95    IER=1
      GO TO 20
100   DO 105 L=1,NXX
      MT=KJ1-L+1
      TEMP=COEF(MT)
      COEF(MT)=WORK(L)
105   WORK(L)=TEMP
      ITEMP=N
      N=NX
      NX=ITEMP
      IF(IFIT) 120,55,120
110   IF(IFIT) 115,50,115
115   X=XPR
      Y=YPR
120   IFIT=0
122   IF(ABS(Y/X)-1.0E-04) 135,125,125
125   ALPHA=X+X
      SUMSQ=X*X+Y*Y
      N=N-2
      GO TO 140
130   X=0.0
      NX=NX-1
      NXX=NXX-1
135   Y=0.0
      SUMSQ=0.0
      ALPHA=X
      N=N-1
140   WORK(2)=WORK(2)+ALPHA*WORK(1)
145   DO 150 L=2,N
150   WORK(L+1)=WORK(L+1)+ALPHA*WORK(L)-SUMSQ*WORK(L-1)
155   ROOTR(N2)=X
      ROOTI(N2)=Y
      N2=N2+1
      IF(SUMSQ) 160,165,160
160   Y=-Y
      SUMSQ=0.0
      GO TO 155
165   IF (N) 20,20,45
20    RETURN
      END
```

```
C
C
C     THIS PROGRAM FINDS THE LEAST SQUARE VALUE
C
C     K ... NO. OF TRANSFER FUNCTION.
C     N ... DEGREE OF DENOMINATOR.
C     THE COEFFICIENTS ARE READ IN ASCENDING ORDER.
C
C
      DOUBLE PRECISION D(20),C(20)
      READ(5,10) K
      DO 5 J=1,K
      WRITE(6,30)J
   30 FORMAT(////10X,'LEAST SQUARE VALUE OF SET',I2)
      DO 5 I=1,2
      READ(5,10) N
      ND=N+1
      READ(5,20) (D(L),L=1,ND)
      READ(5,20) (C(M),M=1,N)
   20 FORMAT((4D20.8))
   10 FORMAT(I2)
      CALL ISE(N,D,C)
    5 CONTINUE
      RETURN
      END


      SUBROUTINE ISE(N,D,C)
      DOUBLE PRECISION A(20),B(20),C(20),D(20),G(40),DD(20,20),
     1DN(20,20),T(2,20),F(20,20)
      N1=N+1
      WRITE(6,600) (D(I),I=1,N1)
      WRITE(6,600) (C(I),I=1,N)
      NX=N
      DO 50 I=1,2
      DO 51 J=1,N
      NJ=NX-2*(J-1)
      IF(NJ.LE.0) T(I,J)=0.
51    IF(NJ.GT.0) T(I,J)=D(NJ)
50    NX=N+1
      DO 60 I=1,N
      DO 60 J=1,N
      DD(I,J)=0.
60    DN(I,J)=0.
      L=0
      LL=0
      LJ=1
      DO 70 I=1,N
      L=L+1
      IF(L.EQ.3) LJ=LJ+1
      IF(L.EQ.3) L=1
      DO 71 J=LJ,N
      LL=LL+1
      DD(I,J)=T(L,LL)
71    DN(I,J)=T(L,LL)
70    LL=0
      N2=2*N
      DO 80 I=1,N2
80    G(I)=0.
      L=0
      LL=0
      DO 81 I=1,N
      DO 82 J=1,N
      L=LL+J
 82   G(L)=C(J)*(-1.)**(J+1)*C(I)+G(L)
81    LL=LL+1
      DO 90 J=1,N
      JJ=2*(N-J)+1
90    DN(I,J)=G(JJ)
```

```
      CALL INVER(DN,N,E,0,DETN)
      CALL INVER(DD,N,E,0,DETD)
      XI=(-1.)**(N-1)/(2.*D(N1))*DETN/DETD
      WRITE(6,600) XI
600   FORMAT(//(5E20.12///))
601   FORMAT(/1X,8E14.6/)
      RETURN
      END



      SUBROUTINE INVER  (A,N,B,M,DET)
      DOUBLE PRECISION A(20,20),B(20,20),IPVOT(20),INDEX(20,2),
     1PIVOT(20),DET,T
      EQUIVALENCE (IROW,JROW),(ICOL,JCOL)
   57 DET=1.
      DO 17 J=1,N
   17 IPVOT(J)=0
      DO 135 I=1,N
      T=0.
      DO 9 J=1,N
      IF(IPVOT(J)-1) 13,9,13
   13 DO 23 K=1,N
      IF(IPVOT(K)-1) 43,23,81
   43 IF (DABS(T)-DABS(A(J,K))) 83,23,23
   83 IROW=J
      ICOL=K
      T=A(J,K)
   23 CONTINUE
    9 CONTINUE
          IPVOT(ICOL)=IPVOT(ICOL)+1
      IF(IROW-ICOL) 73,109,73
   73 DET=-DET
      DO 12 L=1,N
      T=A(IROW,L)
      A(IROW,L)=A(ICOL,L)
   12 A(ICOL,L)=T
      IF(M) 109,109,33
   33 DO 2 L=1,M
      T=B(IROW,L)
      B(IROW,L)=B(ICOL,L)
    2 B(ICOL,L)=T
  109 INDEX(I,1)=IROW
      INDEX(I,2)=ICOL
      PIVOT(I)=A(ICOL,ICOL)
      DET=DET*PIVOT(I)
      A(ICOL,ICOL)=1.
      DO 205 L=1,N
  205 A(ICOL,L)=A(ICOL,L)/PIVOT(I)
      IF(M) 347,347,66
   66 DO 52 L=1,M
   52 B(ICOL,L)=B(ICOL,L)/PIVOT(I)
  347 DO 134 LI=1,N
      IF (LI-ICOL) 21,134,21
   21 T=A(LI,ICOL)
      A(LI,ICOL)=0.
      DO 89 L=1,N
   89 A(LI,L)=A(LI,L)-A(ICOL,L)*T
      IF(M) 134,134,18
   18 DO 68 L=1,M
   68 B(LI,L)=B(LI,L)-B(ICOL,L)*T
  134 CONTINUE
  135 CONTINUE
  222 DO 3 I=1,N
      L=N-I+1
```

```
      IF(INDEX(L,1)-INDEX(L,2)) 19,3,19
   19 JROW=INDEX(L,1)
      JCOL=INDEX(L,2)
      DO 549 K=1,N
      T=A(K,JROW)
      A(K,JROW)=A(K,JCOL)
      A(K,JCOL)=T
  549 CONTINUE
    3 CONTINUE
   81 RETURN
      END
```

```
C
C
C  THIS PROGRAM FINDS THE QUOTIENTS OF A SINGLE-INPUT/SINGLE-OUTPUT
C  SYSTEM.
C  L ... DEGREE OF THE CHARACTERISTIC EQUATION.
C  BOTH DENOMINATOR AND NUMERATOR POLYNOMIALS ARE READ IN
C  ASCENDING ORDER.
C
C
      DOUBLE PRECISION A(20,20),H(40)
      DO 1 I=1,20
      DO 1 J=1,20
    1 A(I,J)=0.0
      READ(5,5)L
      N=L+1
      READ(5,10)(A(1,I),I=1,N)
      READ(5,10)(A(2,I),I=1,L)
    5 FORMAT(I2)
   10 FORMAT((4D20.6))
      K=0
      LL=L+L
      DO 20 I=1,LL
      H(I)=A(I,1)/A(I+1,1)
      WRITE(6,701)I,H(I)
  701 FORMAT(//10X,'H(',I2,') =',D20.6)
      IF(I.EQ.LL) GO TO 3
      K=K+1
      IF(K.EQ.2) K=0
      IF(K.EQ.1) N=N-1
    3 CONTINUE
      M=I+2
      DO 30 J=1,N
      A(M,J)=A(M-2,J+1)-H(M-2)*A(M-1,J+1)
   30 CONTINUE
   20 CONTINUE
   33 CONTINUE
      STOP
      END
```

```
C
C  THIS PROGRAM FINDS THE APPROXIMATED NUMERATOR POLYNOMIAL OF A
C  SINGLE-INPUT/SINGLE-OUTPUT SYSTEM.
C
C  L ... DEGREE OF THE CHARACTERISTIC EQUATION.
C  THE QUOTIENTS AND THE COEFFICIENTS OF THE DENOMINATOR POLYNOMIAL
C  ARE READ IN ASCENDING ORDER.
C
C
      DOUBLE PRECISION A(10,10),H(10)
      READ(5,5)L
      READ(5,10)(H(I),I=1,L)
      LP1=L+1
      READ(5,10)(A(1,I),I=1,LP1)
    5 FORMAT(I2)
   10 FORMAT((4D20.6))
      DO 1 I=1,L
    1 A(I+1,1)=A(I,1)/H(I)
      L1=L-1
      DO 2 J=1,L1
      LJ=L-J
      DO 2 I=1,LJ
    2 A(I+1,J+1)=(A(I,J+1)-A(I+2,J))/H(I)
      DO 200 I=1,L
      WRITE(6,105)I,A(2,I)
  105 FORMAT(10X,'A(2,',I2,') =',D20.8)
  200 CONTINUE
      STOP
      END
```

```
C THIS PROGRAM WILL FIND THE MATRIX CONTINUED FRACTION
C EXPANSION AND INVERSION OF THREE CAUER FORMS. IT WILL
C ACCEPT AS ITS INPUT, STATE EQUATIONS, TRANSFER FUNCTION
C MATRICES OR MATRIX QUOTIENTS.
      DOUBLE PRECISION A(21,21),B(21,21),CT(21,21),AA(20),
     *H(3,33),
     *E(3,33),HS(3,60),HH(3,30),HP(3,30),DETN
 1000 READ(5,3,END=99)N,M,KAPPR,K1,KK
    3 FORMAT(5I5)
C N...ORDER OF ORIGINAL SYSTEM
C M...DIMENSION OF MATRICES
C KAPPR...ORDER OF APPROXIMATION
C IF K1=1...READ STATE EQUATION
C IF K1=2...READ TRANSFER FUNCTION
C IF K1=3...READ MATRIX QUOTIENTS
C IF KK=1...FIRST CAUER EXPANSION AND INVERSION
C IF KK=2...SECOND CAUER EXPANSION AND INVERSION
C IF KK=3...THIRD CAUER EXPANSION AND INVERSION
      WRITE(6,13)
   13 FORMAT(1H1)
      WRITE(6,10)N,M,KAPPR
   10 FORMAT(' ORDER OF ORIGINAL SYSTEM...',I2,/,' DIMENSION
     *OF MATRIC',
     *'ES...',I2,/,' ORDER OF APPROXIMATION...',I2)
      N1=N+1
      MN=M*N
      MN1=M*N1
      MN2=MN/2
      K2=KAPPR/2
      KDIM=M*K2
C FORM B ARRAY
      DO 6 I=1,K2
      JJ=(I-1)*M
      DO 6 J=1,M
      DO 6 K=1,M
      B(J+JJ,K)=0.
    6 IF(J.EQ.K)B(J+JJ,K)=1.
      IF(K1.EQ.2)GO TO 8
      IF(K1.NE.3)GO TO 30
      GO TO (111,222,333),KK
C READ IN STATE EQUATION DX/DT=AX+BU AND Y=CTX
   30 DO 2 I=1,N
    2 READ(5,1)(A(I,J),J=1,N)
    1 FORMAT((4D20.8))
      DO 4 I=1,N
    4 READ(5,1)(B(I,J),J=1,M)
      DO 5 I=1,M
    5 READ(5,1)(CT(I,J),J=1,N)
      CALL FADDV(M,N,M,A,B,CT,AA,E)
C FORM H ARRAY
      DO 11 I=1,M
      DO 11 J=1,MN1
   11 H(I,J)=0.
      DO 12 J=1,M
      DO 12 K=1,N1
      IJ=M*(K-1)
      II=N1-K
      IF(II)15,15,14
   15 H(J,J+IJ)=1.
      GO TO 12
   14 H(J,J+IJ)=AA(II)
   12 CONTINUE
C FORM B ARRAY
      DO 90 I=1,K2
      JJ=(I-1)*M
      DO 90 J=1,M
      DO 90 K=1,M
      B(J+JJ,K)=0.
   90 IF(J.EQ.K)B(J+JJ,K)=1.
      GO TO (100,200,300),KK
```

```
C READ IN TRANSFER FUNCTION ROW BY ROW DENOMINATOR FIRST
C NUMERATOR IS MXM*(N+1)
C DENOMINATOR IS MXM*(N+1)
C NOTE: COEFFICIENT MATRICES ARE ARRANGED IN ASCENDING ORDER
C S**0.....S**N, S**N COEFFICIENT IN NUMERATOR MUST BE INPUT
C AS ZEROES.
    8 DO 16 I=1,M
   16 READ(5,1)(H(I,J),J=1,MN1)
      DO 17 I=1,M
   17 READ(5,1)(E(I,J),J=1,MN1)
      GO TO (100,200,300),KK
C **** FIRST CAUER EXPANSION AND INVERSION ****
  100 WRITE(6,101)
  101 FORMAT(//,1X,20('*'),' FIRST MATRIX CAUER FORM ',20('*'
     *),//)
      DO 18 I=1,N1
      II=(I-1)*M
      IN=(N1-I)*M
      IN1=IN-M
      DO 19 J=1,M
      DO 19 K=1,M
      E(J,K+IN)=H(J,K+II)
      IF(IN1)20,21,21
   20 H(J,K+II)=0.
      GO TO 19
   21 H(J,K+II)=E(J,K+IN1)
   19 CONTINUE
   18 CONTINUE
      CALL SCAUER(N,M,E,H,HS)
      GO TO 23
  111 WRITE(6,101)
      DO 24 I=1,M
      READ(5,1)(HS(I,J),J=1,MN)
   24 WRITE(6,31)(HS(I,J),J=1,MN)
   31 FORMAT(//(1X,6E19.8))
   23 CALL MATG(M,KAPPR,HS,A)
      CALL INVRE(A,KDIM,CT,0,DETN)
      CALL MALTR(KDIM,KDIM,A,M,B,CT)
      DO 22 I=1,KDIM
      DO 22 J=1,M
   22 CT(I,J)=-CT(I,J)
C FORM CT ARRAY
      CALL TCAUER(N,M,H,E,HH,HP)
      GO TO 25
  333 WRITE(6,301)
      DO 26 I=1,M
      READ(5,1)(HH(I,J),J=1,MN2)
   26 WRITE(6,31)(HH(I,J),J=1,MN2)
      DO 27 I=1,M
      READ(5,1)(HP(I,J),J=1,MN2)
   27 WRITE(6,31)(HP(I,J),J=1,MN2)
   25 CALL FRANK(KAPPR,M,HH,HP,H,E)
      N=K2
      GO TO 202
C **** SECOND CAUER EXPANSION AND INVERSION ****
  200 WRITE(6,201)
  201 FORMAT(//,1X,20('*'),' SECOND MATRIX CAUER FORM ',20(
     *'*'),//)
      DO 7 I=2,KAPPR,2
      KK1=((I-2)/2)*M
      KK2=(I-1)*M
      DO 7 J=1,M
      DO 7 K=1,M
    7 B(J,K+KK1)=HS(J,K+KK2)
      WRITE(6,13)
      CALL FADDV(M,KDIM,M,A,CT,B,AA,H)
      GO TO 1000
C **** THIRD CAUER EXPANSION AND INVERSION ****
  300 WRITE(6,301)
  301 FORMAT(//,1X,20('*'),' THIRD MATRIX CAUER FORM ',20(
     *'*'),//)
```

```
  202 CALL SCALER(N,M,H,E,HS)
      GO TO 28
  222 WRITE(6,201)
      DO 29 I=1,M
      READ(5,1)(HS(I,J),J=1,MN)
   29 WRITE(6,31)(HS(I,J),J=1,MN)
   28 CALL MATG(M,KAPPR,HS,A)
C FORM CT ARRAY
      DO 70 I=2,KAPPR,2
      KK1=((I-2)/2)*M
      KK2=(I-1)*M
      DO 70 J=1,M
      DO 70 K=1,M
   70 CT(J,K+KK1)=HS(J,K+KK2)
      WRITE(6,13)
      CALL FADDV(M,KDIM,M,A,B,CT,AA,H)
      GO TO 1000
   99 STOP
      END



      SUBROUTINE FADDV(L,N,M,A,B,CT,D,Q)
      DOUBLE PRECISION A(21,21),B(21,21),CT(21,21),R(21,21),
     *C(21,21),
     *DD(21,21),DI(21,21),D(20),TRACE,FII,Q(3,33)
C **** THE LEVERRIER ALGORITHM
C **** INPUT IS DX/DT=AX+BU AND Y=CTX
C **** A MATRIX IS NXN
C **** B MATRIX IS NXM
C **** CT MATRIX IS LXN
      WRITE(6,2)
    2 FORMAT(50X,'THE A MATRIX IS',///)
      DO 3 I=1,N
    3 WRITE(6,5)(A(I,J),J=1,N)
    5 FORMAT(//(3X,(6E18.8)))
      WRITE(6,6)
    6 FORMAT(///,50X,'THE B MATRIX IS',///)
      DO 7 I=1,N
    7 WRITE(6,5)(B(I,J),J=1,M)
      WRITE(6,8)
    8 FORMAT(///,50X,'THE CT MATRIX IS',///)
      DO 9 I=1,L
    9 WRITE(6,5)(CT(I,J),J=1,N)
      DO 10 I=1,N
      DO 10 J=1,N
      R(I,J)=0.
      IF(I.EQ.J)R(I,J)=1.
   10 CONTINUE
      KN=N*M
      DO 50 I=1,L
      DO 50 J=1,M
   50 Q(I,J+KN)=0.
      DO 11 II=1,N
      CALL MALTR(N,N,R,N,A,DI)
      TRACE=0.
      DO 12 I=1,N
   12 TRACE=TRACE+DI(I,I)
      FII=II
      D(II)=-TRACE/FII
      CALL MALTR(N,N,R,M,B,C)
      CALL MALTR(L,N,CT,M,C,DD)
      WRITE(6,21)II
   21 FORMAT(//,5X,'THE Q(',I2,') MATRIX IS')
      DO 14 I=1,L
   14 WRITE(6,20)(DD(I,J),J=1,M)
   20 FORMAT(//(3X,(6E19.8)))
      KN=(N-II)*M
      DO 30 I=1,L
      DO 30 J=1,M
   30 Q(I,J+KN)=DD(I,J)
```

```
      DO 16 I=1,N
      DO 16 J=1,N
      R(I,J)=DI(I,J)
      IF(I.EQ.J)R(I,J)=R(I,J)+D(II)
   16 CONTINUE
   11 CONTINUE
      WRITE(6,17)
   17 FORMAT(////,2X,'THE COEFFICIENTS OF THE CHAR. EQ.
     *ARE',/)
      DO 18 I=1,N
   18 WRITE(6,19)I,D(I)
   19 FORMAT(2X,'A(',I2,')=',E19.8)
      RETURN
      END
```

```
      SUBROUTINE TCALER(N,M,A,B,HH,HP)
      DOUBLE PRECISION A(3,33),B(3,33),C(3,27),E(3,3),F(3,3)
     *,H(3,3),
     *P(3,3),X(3,3),Y(3,3),DETN,HH(3,30),HP(3,30)
C  A...DENOMINATOR MATRICES
C  M....DIMENSION OF MATRICES
CTHIS PROGRAM EXPANDS A RATIONAL TRANSFER FUNCTION INTO A
CCONTINUED FRACTION EXPANSION OF THE THIRD MATRIX CAUER FORM.
      WRITE(6,3)N,M
    3 FORMAT(' ORDER OF CHARACTERISTIC EQUATION IS ',I3,//,'
     *DIMENSION',
     *' OF MATRIX IS ',I3,/)
      MN1=M*(N+1)
      MN2=M*N
      DO 4 I=1,M
    4 WRITE(6,6)(A(I,J),J=1,MN1)
    6 FORMAT(//(1X,6E19.8))
      DO 79 I=1,M
   79 WRITE(6,6)(B(I,J),J=1,MN2)
      DO 100 K=1,N
      DO 20 I=1,M
      DO 20 J=1,MN1
   20 C(I,J)=0.
      DO 30 I=1,M
      DO 30 J=1,M
      E(I,J)=A(I,J)
   30 F(I,J)=B(I,J)
      CALL INVER(F,M,X,0,DETN)
      CALL MALTP(M,M,E,M,F,H)
      WRITE(6,8)K
    8 FORMAT(/' THE H(',I2,') MATRIX IS')
      DO 40 I=1,M
   40 WRITE(6,9)(H(I,J),J=1,M)
    9 FORMAT(//(1X,6E19.8))
      LB=(N-K)*M
      LA=LB+M
      DO 50 I=1,M
      DO 50 J=1,M
      E(I,J)=A(I,J+LA)
   50 F(I,J)=B(I,J+LB)
      CALL INVER(F,M,X,0,DETN)
      CALL MALTP(M,M,E,M,F,P)
      WRITE(6,10)K
   10 FORMAT(/,' THE H(',I2,') PRIME MATRIX IS')
      DO 61 I=1,M
   61 WRITE(6,9)(P(I,J),J=1,M)
```

```
      KL=(K-1)*M
      DO 60 I=1,M
      DO 60 J=1,M
      HH(I,J+KL)=H(I,J)
   60 HP(I,J+KL)=P(I,J)
      IF(K.EQ.N)GO TO 1
      LL=N-K
      DO 70 L=1,LL
      ML=M*L
      ML1=(L-1)*M
      DO 80 I=1,M
      DO 80 J=1,M
      E(I,J)=B(I,J+ML)
   80 F(I,J)=B(I,J+ML1)
      CALL MALTP(M,M,H,M,E,X)
      CALL MALTP(M,M,P,M,F,Y)
      DO 90 I=1,M
      DO 90 J=1,M
   90 C(I,J+ML1)=A(I,J+ML)-X(I,J)-Y(I,J)
   70 CONTINUE
      WRITE(6,120)
  120 FORMAT(//,10X,'THE ROUTH ARRAY IS')
      DO 200 I=1,M
  200 WRITE(6,9)(C(I,J),J=1,ML)
      DO 250 I=1,M
      DO 400 J=1,LA
  400 A(I,J)=B(I,J)
      DO 500 J=1,LB
  500 B(I,J)=C(I,J)
  250 CONTINUE
  100 CONTINUE
    1 RETURN
      END


      SUBROUTINE FRANK(KAPPR,M,H,HP,D,E)
C  THIS ROUTINE FINDS THE INVERSION OF A THIRD MATRIX CAUER
   EXPANSION
      DOUBLE PRECISION H(3,30),HP(3,30),C(3,33),D(3,33),E(3,
     * 33),X(3,3),
     *Y(3,3),X1(3,3),Y1(3,3),X2(3,3),Y2(3,3)
C   KAPPR....NO. OF H'S PLUS NO. OF HP'S
C   M....DIMENSION OF H'S AND HP'S
      K2=KAPPR/2
      K22=(K2+1)*M
      DO 20 I=1,M
      DO 20 J=1,K22
      D(I,J)=0.
      E(I,J)=0.
   20 IF(I.EQ.J)E(I,J)=1.
      KL=(K2-1)*M
      DO 30 J=1,M
      DO 30 K=1,M
      D(J,K)=H(J,K+KL)
   30 D(J,K+M)=HP(J,K+KL)
      IF(K2.EQ.1)RETURN
      DO 40 I=2,K2
      KM2=(K2-I)*M
      KM1=(I-1)*M
      KN=I*M
      DO 50 J=1,M
      DO 50 K=1,M
      X(J,K)=H(J,K+KM2)
      Y(J,K)=HP(J,K+KM2)
      X1(J,K)=D(J,K)
   50 Y1(J,K)=D(J,K+KM1)
      CALL MALTP(M,M,X,M,X1,X2)
      CALL MALTP(M,M,Y,M,Y1,Y2)
```

```
      DO 60 J=1,M
      DO 60 K=1,M
      C(J,K)=X2(J,K)
   60 C(J,K+KN)=Y2(J,K)
      KK1=I-1
      DO 70 II=1,KK1
      KJ=II*M
      KJ1=(II-1)*M
      DO 80 J=1,M
      DO 80 K=1,M
      X1(J,K)=D(J,K+KJ1)
   80 Y1(J,K)=D(J,K+KJ)
      CALL MALTP(M,M,Y,M,X1,X2)
      CALL MALTP(M,M,X,M,Y1,Y2)
      DO 90 J=1,M
      DO 90 K=1,M
   90 C(J,K+KJ)=X2(J,K)+Y2(J,K)+E(J,K+KJ1)
   70 CONTINUE
      KN1=(I+1)*M
      DO 91 J=1,M
      DO 92 K=1,KN
   92 E(J,K)=D(J,K)
      DO 93 K=1,KN1
   93 D(J,K)=C(J,K)
   91 CONTINUE
   40 CONTINUE
      RETURN
      END


      SUBROUTINE SCALER(N,M,A,B,HH)
      DOUBLE PRECISION A(3,33),B(3,33),C(3,33),E(3,3),F(3,3)
     *,G(3,3),
     *H(3,3),DETN,HH(3,60)
      WRITE(6,605) N,M
605   FORMAT(10X,'  N...',I2,'  M...',I2)
      N1=N+1
      MN1=M*N1
      N2=N*2
      KN=N
      KMN=MN1
      DO 10 I=1,M
10    WRITE (6,602) (A(I,J),J=1,MN1)
602   FORMAT (//(1X,6E19.8))
      DO 20 I=1,M
20    WRITE(6,602) (B(I,J),J=1,MN1)
      KS=1
      DO 100 K=1,N2
      DO 30 I=1,M
      DO 30 J=1,MN1
30    C(I,J)=0.
      DO 40 I=1,M
      DO 40 J=1,M
      E(I,J)=A(I,J)
40    F(I,J)=B(I,J)
      CALL INVER (F,M,G,0,DETN)
      CALL MALTP (M,M,E,M,F,H)
      WRITE (6,603) K
603   FORMAT (///5X,19HTHE REQUIRED   H ( ,I2,5H ) IS//)
      KSS=(K-1)*M
      DO 50 I=1,M
      WRITE(6,602) (H(I,J),J=1,M)
      DO 50 J=1,M
   50 HH(I,J+KSS)=H(I,J)
      IF (K.EQ.N2)RETURN
      IF (KS.EQ.2) KN=KN-1
      DO 200 L=1,KN
      ML=M*L
      ML1=(L-1)*M
```

```
      DO 6C I=1,M
      DO 60 J=1,M
60    F(I,J)=B(I,J+ML)
      CALL MALTP (M,M,H,M,F,E)
      DO 70 I=1,M
      DO 70 J=1,M
70    C(I,J+ML1)=A(I,J+ML)-E(I,J)
200   CONTINUE
      WRITE (6,609)
 609  FORMAT (//10X,'THE  ROUTH ARRAY  IS')
      DO 888  I=1,M
888   WRITE(6,606) (C(I,J),J=1,ML)
606   FCRMAT (//2X,6E19.8)
      IF (KS.EQ.2) KMN=KMN-M
      DO 80 I=1,M
      DC 80 J=1,KMN
      A(I,J)=B(I,J)
80    B(I,J)=C(I,J)
      KKS=KS
      IF(KKS.EQ.2) KS=1
      IF(KKS.EQ.1) KS=2
100   CCNTINUE
      RETURN
      END


      SUBROUTINE MATG(N,M,HS,HA)
      DOUBLE PRECISION HS(3,60),HA(21,21),HTT(3,3),HTS(3,3)
     *,HT(3,3)
      NM=N*M/2
      M2=M/2
      DO 70 I=1,NM
      DC 70 J=1,NM
70    HA(I,J)=0.
      DO 5C IT=1,M2
      KL=(IT-1)*N
      KLL=(2*IT-1)*N
      DO 51 I=1,N
      DO 51 J=1,N
51    HT(I,J)=HS(I,KLL+J)
      DO 53 K=1,IT
      KK=2*(K-1)*N
      DO 52 I=1,N
      DC 52 J=1,N
52    HTT(I,J)=HS(I,KK+J)
      CALL MALTP (N,N,HTT,N,HT,HTS)
      DC 55 L=K,M2
      LL=(L-1)*N
      DO 54 I=1,N
      DO 54 J=1,N
54    HA(LL+I,KL+J)=HA(LL+I,KL+J)-HTS(I,J)
55    CONTINUE
53    CCNTINUE
50    CCNTINUE
      RETURN
      END
```

```
      SUBROUTINE INVER  (A,N,B,M,DET)
      DOUBLE PRECISION A(3,3),B(3,3),IPVOT(3),INDEX(3,2),
     *PIVOT(3),DET,T,
     *XX
      EQUIVALENCE (IROW,JROW),(ICOL,JCOL)
      DET=1.
      DO 17 J=1,N
   17 IPVOT(J)=0
      DO 135 I=1,N
      T=0.
      DO 9 J=1,N
      IF(IPVOT(J)-1) 13,9,13
   13 DO 23 K=1,N
      IF(IPVOT(K)-1) 43,23,81
   43 IF (DABS(T)-DABS(A(J,K))) 83,23,23
   83 IROW=J
      ICOL=K
      T=A(J,K)
   23 CONTINUE
    9 CONTINUE
      IPVOT(ICOL)=IPVOT(ICOL)+1
      IF(IROW-ICOL) 73,109,73
   73 DET=-DET
      DO 12 L=1,N
      T=A(IROW,L)
      A(IROW,L)=A(ICOL,L)
   12 A(ICOL,L)=T
      IF(M) 109,109,33
   33 DO 2 L=1,M
      T=B(IROW,L)
      B(IROW,L)=B(ICOL,L)
    2 B(ICOL,L)=T
  109 INDEX(I,1)=IROW
      INDEX(I,2)=ICOL
      PIVOT(I)=A(ICOL,ICOL)
      DET=DET*PIVOT(I)
      A(ICOL,ICOL)=1.
      DO 205 L=1,N
  205 A(ICOL,L)=A(ICOL,L)/PIVOT(I)
      IF(M) 347,347,66
   66 DO 52 L=1,M
   52 B(ICOL,L)=B(ICOL,L)/PIVOT(I)
  347 DO 134 LI=1,N
      IF (LI-ICOL) 21,134,21
   21 T=A(LI,ICOL)
      A(LI,ICOL)=0.
      DO 89 L=1,N
   89 A(LI,L)=A(LI,L)-A(ICOL,L)*T
      IF(M) 134,134,18
   18 DO 68 L=1,M
   68 B(LI,L)=B(LI,L)-B(ICOL,L)*T
  134 CONTINUE
  135 CONTINUE
      DO 3 I=1,N
      L=N-I+1
      IF(INDEX(L,1)-INDEX(L,2)) 19,3,19
   19 JROW=INDEX(L,1)
      JCOL=INDEX(L,2)
      DO 549 K=1,N
      T=A(K,JROW)
      A(K,JROW)=A(K,JCOL)
      A(K,JCOL)=T
  549 CONTINUE
    3 CONTINUE
   81 XX=10.**(-15)
      IF(DABS(DET).GT.XX)RETURN
      WRITE(6,100)
  100 FORMAT(1X,20('*'),' WARNING, DETERMINANT IS LESS THAN
     1.E-15 ',
```

```
     *20('*'))
      RETURN
      END


      SUBROUTINE INVRE  (A,N,B,M,DET)
      DOUBLE PRECISION A(21,21),B(21,21),IPVOT(21),
     *INDEX(21,2),
     *PIVOT(21),DET,T,XX
      EQUIVALENCE (IROW,JROW),(ICOL,JCOL)
      DET=1.
      DO 17 J=1,N
   17 IPVOT(J)=0
      DO 135 I=1,N
      T=0.
      DO 9 J=1,N
      IF(IPVOT(J)-1) 13,9,13
   13 DO 23 K=1,N
      IF(IPVOT(K)-1) 43,23,81
   43 IF (DABS(T)-DABS(A(J,K))) 83,23,23
   83 IROW=J
      ICOL=K
      T=A(J,K)
   23 CONTINUE
    9 CONTINUE
      IPVOT(ICOL)=IPVOT(ICOL)+1
      IF(IROW-ICOL) 73,109,73
   73 DET=-DET
      DO 12 L=1,N
      T=A(IROW,L)
      A(IROW,L)=A(ICOL,L)
   12 A(ICOL,L)=T
      IF(M) 109,109,33
   33 DO 2 L=1,M
      T=B(IROW,L)
      B(IROW,L)=B(ICOL,L)
    2 B(ICOL,L)=T
  109 INDEX(I,1)=IROW
      INDEX(I,2)=ICOL
      PIVOT(I)=A(ICOL,ICOL)
      DET=DET*PIVOT(I)
      A(ICOL,ICOL)=1.
      DO 205 L=1,N
  205 A(ICOL,L)=A(ICOL,L)/PIVOT(I)
      IF(M) 347,347,66
   66 DO 52 L=1,M
   52 B(ICOL,L)=B(ICOL,L)/PIVOT(I)
  347 DO 134 LI=1,N
      IF (LI-ICOL) 21,134,21
   21 T=A(LI,ICOL)
      A(LI,ICOL)=0.
      DO 89 L=1,N
   89 A(LI,L)=A(LI,L)-A(ICOL,L)*T
      IF(M) 134,134,18
   18 DO 68 L=1,M
   68 B(LI,L)=B(LI,L)-B(ICOL,L)*T
  134 CONTINUE
  135 CONTINUE
      DO 3 I=1,N
      L=N-I+1
      IF(INDEX(L,1)-INDEX(L,2)) 19,3,19
   19 JROW=INDEX(L,1)
      JCOL=INDEX(L,2)
      DO 549 K=1,N
      T=A(K,JROW)
      A(K,JROW)=A(K,JCOL)
      A(K,JCOL)=T
  549 CONTINUE
    3 CONTINUE
```

```
 81 XX=10.**(-15)
    IF(DABS(DET).GT.XX)RETURN
    WRITE(6,100)
100 FORMAT(1X,20('*'),' WARNING, DETERMINANT IS LESS THAN
   *1.E-15 ',
   *20('*'))
    RETURN
    END
```

```
    SUBROUTINE MALTR(N,M,A,L,B,C)
    DOUBLE PRECISION A(21,21),B(21,21),C(21,21),S
    DO 10 I=1,N
    DO 10 J=1,L
    S=0.
    DO 10 K=1,M
    S=S+A(I,K)*B(K,J)
 10 C(I,J)=S
    RETURN
    END
```

```
C
C
C   THIS PROGRAM USES THE VALUES OF MATRIX QUOTIENTS AND
C   APPROXIMATED DENOMINATOR POLYNOMIAL TO DETERMINE THE
C   APPROXIMATED NUMERATOR POLYNOMIAL .
C
C   N ... DEGREE OF APPROXIMATED DENOMINATOR POLYNOMIAL .
C   M ... DIMENSION OF THE MATRIX QUOTIENT.  ,NH=3 .
      DOUBLE PRECISION V(10,10),W(10,10),H(10),DE(10),
     1                 D,LL(10),MM(10)
      READ(5,10)N
      READ(5,20)(H(I),I=1,N)
      READ(5,20)(DE(I),I=1,N)
   10 FORMAT(I2)
   20 FORMAT((4D20.6))
      DO 600 I1=1,N
  600 WRITE(6,601)I1,H(I1)
  601 FORMAT(//10X,'H(',I2,')=',D20.8)
      DO 602 I2=1,N
  602 WRITE(6,603)I2,DE(I2)
  603 FORMAT(//10X,'DE(',I2,')=',D20.8)
      M=N-1
      DO 30 I=1,N
      DO 30 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
   30 CONTINUE
      DO 901 I=1,N
      WRITE(6,701)(V(I,J),J=1,N)
  901 CONTINUE
      DO 40 J=1,N
      DO 507 I5=1,10
      DO 507 J5=1,10
  507 W(I5,J5)=0.0
      J1=J-1
      K1=J+1
      DO 45 L=1,J
   45 W(L,L)=1.0
      IF(K1.GT.N) GO TO 56
      DO 50 K=K1,N
      IND=K-J1
   50 W(K,K)=H(IND)
      IF(K1.GT.M) GO TO 56
      DO 55 K=K1,M
   55 W(K+1,K)=1.0
      DO 801 II=1,N
      WRITE(6,701)(W(II,JJ),JJ=1,N)
  801 CONTINUE
C
   56 CALL GMPRD(V,W,V,N,N,N)
      DO 702 II=1,N
      WRITE(6,701)(V(II,JJ),JJ=1,N)
  702 CONTINUE
   40 CONTINUE
  701 FORMAT((/10X,4D20.6))
C
      CALL GMPRS(V,DE,DE,N)
      WRITE(6,701)(DE(I),I=1,N)
      DO 60 I=1,N
      DO 60 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
   60 CONTINUE
      DO 70 J=1,N
      DO 509 I6=1,10
      DO 509 J6=1,10
  509 W(I6,J6)=0.0
      L1=J-1
      J1=J-1
      IF(L1.EQ.0) GO TO 75
      DO 80 L=1,L1
   80 W(L,L)=1.0
```

```
C
C
C   THIS PROGRAM USES THE VALUES OF MATRIX QUOTIENTS AND
C   APPROXIMATED DENOMINATOR POLYNOMIAL TO DETERMINE THE
C   APPROXIMATED NUMERATOR POLYNOMIAL .
C
C   N ... DEGREE OF APPROXIMATED DENOMINATOR POLYNOMIAL .
C   M ... DIMENSION OF THE MATRIX QUOTIENT.   ,NH=3 .
      DOUBLE PRECISION V(10,10),W(10,10),H(10),DE(10),
     1                 D,LL(10),MM(10)
      READ(5,10)N
      READ(5,20)(H(I),I=1,N)
      READ(5,20)(DE(I),I=1,N)
   10 FORMAT(I2)
   20 FORMAT((4D20.6))
      DO 600 I1=1,N
  600 WRITE(6,601)I1,H(I1)
  601 FORMAT(//10X,'H(',I2,')=',D20.8)
      DO 602 I2=1,N
  602 WRITE(6,603)I2,DE(I2)
  603 FORMAT(//10X,'DE(',I2,')=',D20.8)
      M=N-1
      DO 30 I=1,N
      DO 30 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
   30 CONTINUE
      DO 901 I=1,N
      WRITE(6,701)(V(I,J),J=1,N)
  901 CONTINUE
      DO 40 J=1,N
      DO 507 I5=1,10
      DO 507 J5=1,10
  507 W(I5,J5)=0.0
      J1=J-1
      K1=J+1
      DO 45 L=1,J
   45 W(L,L)=1.0
      IF(K1.GT.N) GO TO 56
      DO 50 K=K1,N
      IND=K-J1
   50 W(K,K)=H(IND)
      IF(K1.GT.M) GO TO 56
      DO 55 K=K1,M
   55 W(K+1,K)=1.0
      DO 801 II=1,N
      WRITE(6,701)(W(II,JJ),JJ=1,N)
  801 CONTINUE
C
   56 CALL GMPRD(V,W,V,N,N,N)
      DO 702 II=1,N
      WRITE(6,701)(V(II,JJ),JJ=1,N)
  702 CONTINUE
   40 CONTINUE
  701 FORMAT((/10X,4D20.6))
C
      CALL GMPRS(V,DE,DE,N)
      WRITE(6,701)(DE(I),I=1,N)
      DO 60 I=1,N
      DO 60 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
   60 CONTINUE
      DO 70 J=1,N
      DO 509 I6=1,10
      DO 509 J6=1,10
  509 W(I6,J6)=0.0
      L1=J-1
      J1=J-1
      IF(L1.EQ.0) GO TO 75
      DO 80 L=1,L1
   80 W(L,L)=1.0
```

```
 75   DO 85 K=J,N
      IND=K-J1
 85   W(K,K)=H(IND)
      M=N-1
      IF(J.GT.M) GO TO 95
      DO 90 K=J,M
 90   W(K+1,K)=1.0
 95   CALL GMPRD(V,W,V,N,N,N)
      DO 1001 II=1,N
      WRITE(6,701)(V(II,JJ),JJ=1,N)
1001  CONTINUE
 70   CONTINUE
C
      CALL MINVR(V,N,10,D,LL,MM)
      WRITE(6,100) D
 100  FORMAT(///20X,'**********WARNING **********'//20X,'D='
     *,D20.8///)
      CALL GMPRS(V,DE,DE,N)
      DO 200 I=1,N
      WRITE(6,105)I,DE(I)
 105  FORMAT(10X,'NU(',I2,') =',D20.8//)
 200  CONTINUE
      STOP
      END
```

```
      SUBROUTINE GMPRD(A,B,C,L,M,N)
      DOUBLE PRECISION A(10,10),B(10,10),C(10,10),
     1        AD(10,10),BD(10,10),CD(10,10)
 100  DO 108 J=1,M
      DO 104 I=1,L
 104  AD(I,J)=A(I,J)
      DO 108 K=1,N
 108  BD(J,K)=B(J,K)
      DO 112 I=1,L
      DO 112 J=1,N
      CD(I,J)=0.0
      DO 112 K=1,M
 112  CD(I,J)=CD(I,J)+AD(I,K)*BD(K,J)
      DO 116 I=1,L
      DO 116 J=1,N
 116  C(I,J)=CD(I,J)
      RETURN
      END
```

```
      SUBROUTINE GMPRS(A,B,C,N)
      DOUBLE PRECISION A(10,10),B(10),C(10),
     1                 AD(10,10),BD(10),CD(10)
      DO 10 I=1,N
      DO 10 J=1,N
 10   AD(I,J)=A(I,J)
      DO 15 I=1,N
 15   BD(I)=B(I)
      DO 20 I=1,N
      CD(I)=0.0
      DO 20 J=1,N
 20   CD(I)=CD(I)+AD(I,J)*BD(J)
      DO 30 K=1,N
 30   C(K)=CD(K)
      RETURN
      END
```

```
      SUBROUTINE MINVR(A,N,ND,D,L,M)
      DOUBLE PRECISION A(1),D,BIGA,HOLD
      DIMENSION L(1),M(1)
      D=1.
      NK=-ND
      DO 80 K=1,N
      NK=NK+ND
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=ND*J-ND
      DO 20 I=K,N
      IJ=IZ+I
10    IF(DABS(BIGA)-DABS(A(IJ)))15,20,20
15    BIGA=A(IJ)
      L(K)=I
      M(K)=J
20    CONTINUE
      J=L(K)
      IF(J-K)35,35,25
25    KI=K-ND
      DO 30 I=1,N
      KI=KI+ND
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
30    A(JI)=HOLD
35    I=M(K)
      IF(I-K) 38,48,38
38    JP=ND*I-ND
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
40    A(JI)=HOLD
48    DO 55 I=1,N
      IF(I-K)50,55,50
50    IK=NK+I
      A(IK)=A(IK)/(-BIGA)
55    CONTINUE
      DO 65 I=1,N
      IK=NK+I
      IJ=I-ND
      DO 65 J=1,N
      IJ=IJ+ND
      IF(I-K)60,65,60
60    IF(J-K)62,65,62
62    KJ=IJ-I+K
      A(IJ)=A(IK)*A(KJ)+A(IJ)
65    CONTINUE
      KJ=K-ND
      DO 75 J=1,N
      KJ=KJ+ND
      IF(J-K)70,75,70
70    A(KJ)=A(KJ)/BIGA
75    CONTINUE
      D=D*BIGA
      A(KK)=1./BIGA
80    CONTINUE
      K=N
100   K=K-1
      IF(K)150,150,105
105   I=L(K)
      IF(I-K)120,120,108
108   JQ=ND*K-ND
      JR=ND*I-ND
```

```
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
110   A(JI)=HOLD
120   J=M(K)
      IF(J-K)100,100,125
125   KI=K-ND
      DO 130 I=1,N
      KI=KI+ND
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
130   A(JI)=HOLD
      GO TO 100
150   IF(DABS(D).LE.1.D-15) GO TO 160
      IF(DABS(D).LT.1.D+15) RETURN
160   WRITE(6,600) D
600   FORMAT(//10X,20('*'),'   WARNING   ',20('*')//
     *20X,'DETERMINANT  =',D20.10/)
      END
```

```
C
C
C THIS PROGRAM FINDS THE APPROXIMATED NUMERATOR POLYNOMIAL OF
C SINGLE-INPUT/SINGLE-OUTPUT SYSTEM BY ANOTHER APPROACH.
C
      DOUBLE PRECISION V(10,10),W(10,10),H(10),DE(10),
     1                 D,LL(10),MM(10)
      READ(5,10)N
      READ(5,20)(H(I),I=1,N)
      READ(5,20)(DE(I),I=1,N)
   10 FORMAT(I2)
   20 FORMAT((4D20.6))
      DO 600 I1=1,N
  600 WRITE(6,601)I1,H(I1)
  601 FORMAT(//10X,'H(',I2,')=',D20.8)
      DO 602 I2=1,N
  602 WRITE(6,603)I2,DE(I2)
  603 FORMAT(//10X,'DE(',I2,')=',D20.8)
      M=N-1
      DO 30 I=1,N
      DO 30 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
   30 CONTINUE
      DO 901 I=1,N
      WRITE(6,701)(V(I,J),J=1,N)
  901 CONTINUE
      DO 40 J=1,N
      CALL SETO(W)
      J1=J-1
      K1=J+1
      DO 45 L=1,J
   45 W(L,L)=1.0
      IF(K1.GT.N) GO TO 56
      DO 50 K=K1,N
      IND=K-J1
   50 W(K,K)=H(IND)
      IF(K1.GT.M) GO TO 56
      DO 55 K=K1,M
   55 W(K+1,K)=1.0
```

```
      DO 801 II=1,N
      WRITE(6,701)(W(II,JJ),JJ=1,N)
 801  CONTINUE
C
  56  CALL GMPRD(V,W,V,N,N,N)
      DO 702 II=1,N
      WRITE(6,701)(V(II,JJ),JJ=1,N)
 702  CONTINUE
  40  CONTINUE
 701  FORMAT((/10X,4D20.6))
C
      CALL GMPRS(V,DE,DE,N)
      WRITE(6,701)(DE(I),I=1,N)
      DO 60 I=1,N
      DO 60 J=1,N
      V(I,J)=0.
      IF(I.EQ.J) V(I,J)=1.0
  60  CONTINUE
      DO 70 J=1,N
      CALL SETO(W)
      L1=J-1
      J1=J-1
      IF(L1.EQ.0) GO TO 75
      DO 80 L=1,L1
  80  W(L,L)=1.0
  75  DO 85 K=J,N
      IND=K-J1
  85  W(K,K)=H(IND)
      M=N-1
      IF(J.GT.M) GO TO 95
      DO 90 K=J,M
  90  W(K+1,K)=1.0
  95  CALL GMPRD(V,W,V,N,N,N)
      DO 1001 II=1,N
      WRITE(6,701)(V(II,JJ),JJ=1,N)
1001  CONTINUE
  70  CONTINUE
C
      CALL MINVR(V,N,10,D,LL,MM)
      WRITE(6,100) D
 100  FORMAT(///20X,'**********WARNING **********'//20X,'D=',
     *D20.8///)
      CALL GMPRS(V,DE,DE,N)
      DO 200 I=1,N
      WRITE(6,105)I,DE(I)
 105  FORMAT(10X,'NU(',I2,') =',D20.8//)
 200  CONTINUE
      STOP
      END
```

```
      SUBROUTINE SETO(W)
      DIMENSION W(10,10)
      DO 10 I=1,10
      DO 10 J=1,10
  10  W(I,J)=0.0
      RETURN
      END
```

```
      SUBROUTINE GMPRD(A,B,C,L,M,N)
      DOUBLE PRECISION A(10,10),B(10,10),C(10,10),
     1         AD(10,10),BD(10,10),CD(10,10)
100   DO 108 J=1,M
      DO 104 I=1,L
104   AD(I,J)=A(I,J)
      DO 108 K=1,N
108   BD(J,K)=B(J,K)
      DO 112 I=1,L
      DO 112 J=1,N
      CD(I,J)=0.0
      DO 112 K=1,M
112   CD(I,J)=CD(I,J)+AD(I,K)*BD(K,J)
      DO 116 I=1,L
      DO 116 J=1,N
116   C(I,J)=CD(I,J)
      RETURN
      END
```

```
      SUBROUTINE GMPRS(A,B,C,N)
      DOUBLE PRECISION A(10,10),B(10),C(10),
     1                 AD(10,10),BD(10),CD(10)
      DO 10 I=1,N
      DO 10 J=1,N
 10   AD(I,J)=A(I,J)
      DO 15 I=1,N
 15   BD(I)=B(I)
      DO 20 I=1,N
      CD(I)=0.0
      DO 20 J=1,N
 20   CD(I)=CD(I)+AD(I,J)*BD(J)
      DO 30 K=1,N
 30   C(K)=CD(K)
      RETURN
      END
```

```
      SUBROUTINE MINVR(A,N,ND,D,L,M)
      DOUBLE PRECISION A(1),D,BIGA,HOLD
      DIMENSION L(1),M(1)
      D=1.
      NK=-ND
      DO 80 K=1,N
      NK=NK+ND
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=ND*J-ND
      DO 20 I=K,N
      IJ=IZ+I
10    IF(DABS(BIGA)-DABS(A(IJ)))15,20,20
15    BIGA=A(IJ)
      L(K)=I
      M(K)=J
20    CONTINUE
      J=L(K)
      IF(J-K)35,35,25
25    KI=K-ND
```

```
      DO 30 I=1,N
      KI=KI+ND
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
30    A(JI)=HOLD
35    I=M(K)
      IF(I-K) 38,48,38
38    JP=ND*I-ND
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
40    A(JI)=HOLD
48    DO 55 I=1,N
      IF(I-K)50,55,50
50    IK=NK+I
      A(IK)=A(IK)/(-BIGA)
55    CONTINUE
      DO 65 I=1,N
      IK=NK+I
      IJ=I-ND
      DO 65 J=1,N
      IJ=IJ+ND
      IF(I-K)60,65,60
60    IF(J-K)62,65,62
62    KJ=IJ-I+K
      A(IJ)=A(IK)*A(KJ)+A(IJ)
65    CONTINUE
      KJ=K-ND
      DO 75 J=1,N
      KJ=KJ+ND
      IF(J-K)70,75,70
70    A(KJ)=A(KJ)/BIGA
75    CONTINUE
      D=D*BIGA
      A(KK)=1./BIGA
80    CONTINUE
      K=N
100   K=K-1
      IF(K)150,150,105
105   I=L(K)
      IF(I-K)120,120,108
108   JQ=ND*K-ND
      JR=ND*I-ND
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
110   A(JI)=HOLD
120   J=M(K)
      IF(J-K)100,100,125
125   KI=K-ND
      DO 130 I=1,N
      KI=KI+ND
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
130   A(JI)=HOLD
      GO TO 100
150   IF(DABS(D).LE.1.D-15) GO TO 160
      IF(DABS(D).LT.1.D+15) RETURN
160   WRITE(6,600) D
600   FORMAT(//10X,20('*'),'    WARNING    ',20('*')//
     *20X,'DETERMINANT  =',D20.10/)
      END
```

```
C
C
C  THIS PROGRAM FINDS THE APPROXIMATED NUMERATOR POLYNOMIAL BY
C  ANOTHER APPROACH.
C
C
      DOUBLE PRECISION A(20,20),B(20,20),C(20,20),
     *                 H(3,60),DE(20,3),D
      DIMENSION LL(20),MM(20)
      READ(5,5) N,M
    5 FORMAT(2I5)
      NH=3
      NA=20
      NB=20
      NC=20
      MN=M*N
      DO 10 I=1,M
      READ(5,1)(H(I,J),J=1,MN)
   10 WRITE(6,15)(H(I,J),J=1,MN)
    1 FORMAT((4D20.8))
   15 FORMAT((//1X,'H',2X,6D19.8))
      DO 20 J=1,M
      READ(5,1)(DE(K,J),K=1,MN)
   20 WRITE(6,25)(DE(K,J),K=1,MN)
   25 FORMAT((//1X,'DE',2X,6D19.8))
      CALL FLO(M,N,H,NH,A,NA,B,NB,C,NC)
      DO 30 I=1,MN
   30 WRITE(6,35)(B(I,J),J=1,MN)
   35 FORMAT((//1X,'LM',2X,6D19.8))
      CALL MULT(B,DE,DE,MN,MN,M)
      CALL FRO(M,N,H,NH,A,NA,B,NB,C,NC)
      CALL MINVR(B,MN,NB,D,LL,MM)
      WRITE(6,40)D
   40 FORMAT(/////10X,'********** WARNING **********'//
     *                8X,'D =',D20.10///)
      DO 50 I=1,MN
   50 WRITE(6,55)(B(I,J),J=1,MN)
   55 FORMAT((//1X,'BI',2X,6D19.8))
      CALL MULT(B,DE,DE,MN,MN,M)
      DO 60 II=1,N
      WRITE(6,65)II
   65 FORMAT(////5X,'THE Q(',I2,') MATRIX IS'/)
      MI=M*(II-1)+1
      MII=M*II
      DO 70 I=MI,MII
   70 WRITE(6,75)(DE(I,J),J=1,M)
   75 FORMAT(//(5X,(6D19.8)))
   60 CONTINUE
      STOP
      END


      SUBROUTINE IDEN(A,NA,N)
      DOUBLE PRECISION A(1)
      K=-NA
      DO 2 J=1,N
      K=K+NA
      DO 1 I=1,N
      KK=K+I
1     A(KK)=0.
      KK=K+J
2     A(KK)=1.
      RETURN
      END
```

```
      SUBROUTINE MULT(A,B,C,L,M,N)
      DOUBLE PRECISION A(20,20),B(20,20),C(20,20),
     *                 AD(20,20),BD(20,20),CD(20,20)
100   DO 108 J=1,M
      DO 104 I=1,L
104   AD(I,J)=A(I,J)
      DO 108 K=1,N
108   BD(J,K)=B(J,K)
      DO 112 I=1,L
      DO 112 J=1,N
      CD(I,J)=0.0
      DO 112 K=1,M
112   CD(I,J)=CD(I,J)+AD(I,K)*BD(K,J)
      DO 116 I=1,L
      DO 116 J=1,N
116   C(I,J)=CD(I,J)
      RETURN
      END

      SUBROUTINE MNR(L,M,N,RR,A,R,NRR,NA,NR)
      DOUBLE PRECISION RR(1),A(1),R(1)
      DO 4 I=1,L
      KKI=I-NR
      KKA=-NA
      DO 4 J=1,N
      KKJ=I-NRR
      KKA=KKA+NA
      KKI=KKI+NR
      R(KKI)=0.
      DO 4 K=1,M
      KKK=KKA+K
      KKJ=KKJ+NRR
4     R(KKI)=R(KKI)+RR(KKJ)*A(KKK)
      RETURN
      END

      SUBROUTINE PICKT(A,NA,B,NB,M,IR,IC)
      DOUBLE PRECISION A(1),B(1)
      L=(IC*NA-NA+IR)*M-M-NA
      LB=-NB
      DO 1 J=1,M
      L=L+NA
      LB=LB+NB
      DO 1 I=1,M
      K=L+I
      N=LB+I
1     B(N)=A(K)
      RETURN
      END

      SUBROUTINE EQM(A,NA,B,NB,L,M)
      DOUBLE PRECISION A(1),B(1)
      K=-NB
      KK=-NA
      DO 1 J=1,M
      K=K+NB
      KK=KK+NA
      DO 1 I=1,L
      N=K+I
      NN=KK+I
1     B(N)=A(NN)
      RETURN
      END
```

```
      SUBROUTINE FLO(M,N,H,NH,A,NA,B,NB,C,NC)
      DOUBLE PRECISION A(1),B(1),C(1),H(1),Y(3,3),Z(3,3)
      NZ=3
      KA=N+1
      MN=M*N
      CALL IDEN(A,NA,MN)
      CALL IDEN(B,NB,MN)
      CALL IDEN(Y,NZ,M)
      DO 10 I=2,N
      IF(I-2)6,5,6
5     CALL PICKT(H,NH,Z,NZ,M,1,I)
      CALL STORE(A,NA,Z,NZ,M,N,N)
      GO TO 8
6     K=KA-I
      DO 7 J=3,I
      K=K+1
      KK=K+1
      CALL PICKT(A,NA,Z,NZ,M,KK,KK)
      CALL STORE(A,NA,Z,NZ,M,K,K)
7     CALL STORE(A,NA,Y,NZ,M,KK,K)
      GO TO 5
8     CALL MMR(MN,MN,MN,A,B,C,NA,NB,NC)
      CALL EQM(C,NC,B,NB,MN,MN)
10    CONTINUE
      END
```

```
      SUBROUTINE FRO(M,N,H,NH,A,NA,B,NB,C,NC)
      DOUBLE PRECISION A(1),B(1),C(1),H(1),X(3,3),Y(3,3)
      NX=3
      I=2
      KA=N
      MN=M*N
      CALL IDEN (A,NA,MN)
      CALL IDEN(B,NB,MN)
      CALL IDEN(Y,NX,M)
      K=N-1
      CALL PICKT(H,NH,X,NX,M,1,1)
      CALL STORE(B,NB,X,NX,M,N,N)
      CALL STORE(A,NA,X,NX,M,K,K)
      CALL STORE(A,NA,Y,NX,M,N,K)
      GO TO 5
3     I=I+1
      K=KA-I
      DO 4 J=2,I
      K=K+1
      KK=K+1
      CALL PICKT(A,NA,X,NX,M,KK,KK)
      CALL STORE(A,NA,X,NX,M,K,K)
4     CALL STORE(A,NA,Y,NX,M,KK,K)
5     CALL PICKT(H,NH,X,NX,M,1,I)
      CALL STORE(A,NA,X,NX,M,N,N)
      CALL MMR(MN,MN,MN,A,B,C,NA,NB,NC)
      CALL EQM(C,NC,B,NB,MN,MN)
      IF(I-N)3,7,3
7     RETURN
      END
```

```
      SUBROUTINE STORE(A,NA,B,NB,M,IR,IC)
      DOUBLE PRECISION A(1),B(1)
      L=(IC*NA-NA+IR)*M-M-NA
      LB=-NB
      DO 1 J=1,M
      L=L+NA
      LB=LB+NB
      DO 1 I=1,M
      K=L+I
      N=LB+I
1     A(K)=B(N)
      RETURN
      END
```