SUBMETRIZABLE SPACES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Laurie Davis Gibson

May 1978

To Emily Davis and Emily Gibson

ACKNOWLEDGEMENTS

I thank Dr. Howard Cook for his guidance and encouragement of my studies in topology. I am fortunate to have had as an advisor a teacher of such skill, one who shared with me so much of his time and his enthusiasm for Mathematics.

I thank those members of the mathematics faculty who have been my teachers. Each has influenced in some way the work that follows.

SUBMETRIZABLE SPACES

An Abstract of a Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Laurie Davis Gibson

May 1978

ABSTRACT

Suppose X is a Moore space. It is known that if X is submetrizable, X has the j-link property for each positive integer j. If X admits a semimetric which is upper semi-continuous and continuous in one variable, then X has a v-normal development, a result due to H. Cook. We prove that if X is separable and X has a v-normal development, then X has the j-link property for each positive integer j. From this follow the corollaries: A Moore-closed space with a v-normal development is compact; and if X is a Moore space with a v-normal development then the closure of every conditionally compact subset of X is compact. We show that if j is an integer greater than 1, there is a Moore space which has the j-link property but not the j+1-link property.

Alster and Przymusiński have defined regular submetrizability and H. Cook has given conditions under which a regularly submetrizable Moore space admits a continuous semimetric. We introduce the stronger notion of normal submetrizability and show that a normally submetrizable space is completely regular. We also prove that if X is a Moore space, X^2 is normally submetrizable, and the diagonal in X^2 is closed in the metric topology, then X is continuously semimetrizable.

TABLE OF CONTENTS

INTRODUCTION

In this dissertation, we examine Moore spaces that are nonmetric but have metric-like properties. Semimetrics offer one indication of how close a Moore space is to being metric: X is a Moore space if, and only if, X is regular and X has an upper semi-continuous semimetric [3]; X is a metric space if X admits a uniformly continuous semimetric [14]. Between Moore and metric spaces lie those spaces with semimetrics that are continuous in one variable, upper semi-continuous and continuous in one variable, and continuous. For this reason, semimetrics appear throughout this dissertation, and, in Section 1, we consider in detail the v-normal development, a concept that arises in semimetrics.

A space X is submetrizable if there is a continuous one-to-one map of X onto a metric space. Thus, if $\mathbf{T}$ is the topology on X, some subcollection of $\mathbf{T}$ is a metric topology. Alster and Przymusiński [1] define regular submetrizability by relating $\mathbf{T}$ and the metric topology more closely. In Section 3, we introduce normal submetrizability, adding a still stronger condition to the relationship. What properties does $\mathbf{T}$ inherit from the metric topology? If X is submetrizable, $\mathbf{T}$ is Hausdorf; if X is regularly submetrizable, $\mathbf{T}$ is regular; and if X is normally submetrizable, we prove that $\mathbf{T}$ is completely regular. In addition, F. G. Slaughter, Jr., observed that if X is a submetrizable space, X has the j-link property for each positive integer j. This property is of interest in the examples of Sections 2 and 4. The examples show that if j > 1, a Moore space with the j-link property need not have the j+1-link property and that there is a Moore space with the j-link property for each positive integer j which is not submetrizable.

Alster and Przymusiński [1] prove that, assuming Martin's Axiom, if X is separable and regularly submetrizable and X is the sum of fewer than c compact sets, then for each positive integer n, X^n is normal. H. Cook [2] also assumes Martin's Axiom to show that a separable Moore space X which is the sum of fewer than c compact sets is regularly submetrizable if, and only if, X is continuously semimetrizable. Since normal submetrizability is stronger than regular submetrizability, we are able, in Theorem 8 of the last section, to relate the former to continuous semimetrizability without extra set-theoretic assumptions. Example T in this section is due to H. Cook. It is included to give insight into normal submetrizability and to show that the hypothesis of Theorem 8 is not a necessary condition for continuous semimetrizability.

SECTION 1.

If X is a topological space, G is a collection of point sets covering X, and O is a point set, we denote by $st_1(O,G) = st(O,G)$, the union of all point sets in G which intersect O; if x is a point, $st(x,G) = st(\{x\},G)$; and if n is a positive integer, $st_{n+1}(O,G) = st_n(st(O,G),G)$. A development for X is a sequence $\{G_n\}_{n=1}^{\infty}$ such that for each positive integer n, G_n is an open cover of X and G_{n+1} refines G_n, and if O is an open set and x is in O, there is a positive integer m such that $st(x,G_m) \subset O$. If for each positive integer n, $G_{n+1} \subset G_n$, then $\{G_n\}_{n=1}^{\infty}$ is a nested development. A Moore space is a regular space which admits a development.

Definition 1. If X is a topological space, j is a positive integer, and $\mathbf{H}$ is a family of open covers of X, the statement that $\mathbf{H}$ has the j-link property means that for each two points p and q of X, there is a cover H in $\mathbf{H}$ such that q is not in $st_j(p,H)$. The space X has the j-link property if there is a countable family $\mathbf{H}$ of open covers of X and $\mathbf{H}$ has the j-link property.

A space has the 1-link property if, and only if, it has a G_δ-diagonal. Every Moore space has the 2-link property. If j is a positive integer, every Moore space with the j-link property has a development that has the j-link property. If a Moore space has the j-link property for every positive integer j, then it has a development having the j-link property for every positive integer j.

In Section 2, we show that if j is a positive integer greater than 1, a Moore space with the j-link property need not have the j+1-link property.

<u>Definition 2</u>. The statement that the development $\{G_n\}_{n=1}^{\infty}$ for the space X is normal means that if n is a positive integer, p and q are points of X, and no element of G_n contains both p and q, then there are open sets O_p and O_q containing p and q respectively such that no element of G_{n+1} intersects both O_p and O_q.

<u>Definition 3</u>. The statement that the development $\{G_n\}_{n=1}^{\infty}$ for the space X is v-normal means that if n is a positive integer and p is a point of X, $\overline{st(p,G_{n+1})} \subset st(p,G_n)$.

A normal development for X is also a v-normal development. A Moore space with a normal development has the 3-link property [2]. Example V_2 in Section 2 is a Moore space with a v-normal development. The space V_2 does not have the 3-link property, thus, it does not have a normal development.

<u>Theorem 1</u>. A separable Moore space with a v-normal development has the j-link property for each positive integer j.

Proof. Suppose X is a Moore space, $\{G_n\}_{n=1}^{\infty}$ is a v-normal development for X, $\{a_n | n \text{ is a positive integer}\}$ is a dense subset of X, and j is a positive integer greater than 1. Suppose each of k and n is a positive integer and define H_{kn} to be the collection of all sets b_i^{kn} where $1 \leq i \leq j+1$, $b_1^{kn} = st(a_k, G_{n+j-1})$, $b_{j+1}^{kn} = X - \overline{st(a_k, G_{n+1})}$, and if $1 \leq m \leq j+1$, $b_m^{kn} = st(a_k, G_{n+j-m}) - \overline{st(a_k, G_{n+j-m+2})}$.

If x is in X and x is not in b_1^{kn}, let m be the least positive integer i such that x is not in $st(a_k, G_i)$. If $m \leq n$, $x \in X - st(a_k, G_n) \subset X - \overline{st(a_k, G_{n+1})} = b_{i+1}^{kn}$. If $n < m$, then $x \in st(a_k, G_{m-1}) - \overline{st(a_k, G_{m+1})} = b_{n+j-m+1}^{kn}$. Thus, H_{kn} covers X.

Suppose i is a positive integer. If $1 < i < j+1$, $b_i^{kn} \subset st(a_k, G_{n+j-1})$; if $1 \leq i \leq j+1$, $b_i^{kn} \cap st(a_k, G_{n+j-i+2}) = \phi$; and if p is a positive integer, $1 \leq p \leq i-2$, then $b_i^{kn} \cap st(a_k, G_{n+j-p}) = \phi$, and $b_i^{kn} \cap b_p^{kn} = \phi$. It follows that if x is in $st(a_k, G_{n+j})$ and if q is a positive integer, $1 \leq q \leq j+1$, then $st_q(x, H_{kn}) \subset \bigcup_{i=1}^{q} b_i^{kn}$. Thus, $st_j(x, H_{kn}) \subset st(a_k, G_n)$. Define **H** to be the collection of all H_{kn} where each of k and n is a positive integer.

Suppose x and y are points of X. There is a positive integer n such that $st(x, G_n) \cap st(y, G_n) = \phi$, and a positive integer k such that a_k is in $st(x, G_{n+j})$. Then, x is in $st(a_k, G_{n+j})$ and $st_j(x, H_{kn}) \subset st(a_k, G_n)$. Since a_k is not in $st(y, G_n)$, y is not in $st(a_k, G_n)$ and, therefore, y is not in $st_j(x, H_{kn})$. **H** has the j-link property.

H is countable, so X has the j-link property.

Definition 4. A topological space X is Moore-closed if X is a Moore space and X is closed in every Moore space in which it is embedded.

A subset M of a space X is conditionally compact provided that every infinite subset of M has a limit point in X. There exists a Moore space with a conditionally compact subset whose closure is not compact [11, p. 66]. J. W. Green [4] has shown that a noncompact Moore space that has a dense conditionally compact subset is Moore-closed. A Moore-closed space X with the 3-link property is compact [5]. From this it follows, if a Moore space X has the 3-link property, then every conditionally compact subset of X has a compact closure [6]. G. M. Reed [12] proved that every Moore-closed space is separable. Then, it follows from Theorem 1 that a Moore-closed space with a v-normal development has the 3-link property and so, is compact. A second proof of this statement is included because we feel it will give the reader a better understanding of the v-normal development in Moore spaces.

Lemma. Suppose X is a Moore space, $\{z_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ are sequences such that, if n is a positive integer, U_n is an open set, z_n is in U_n, and $U_{n+1} \subset U_n$. If no sequence $\{y_n\}_{n=1}^{\infty}$ such that, for each positive integer n, y_n is in U_n, has a cluster point, then there is an embedding f of X into a Moore space in which $\{f(z_n)\}_{n=1}^{\infty}$ converges.

Proof. Suppose $\{G_n\}_{n=1}^{\infty}$ is a nested development for X. If m is a positive integer, there is a sequence $\{D_n^m\}_{n=1}^{\infty}$ with the property that, if n is a positive integer, $z_n \in D_n^m \in G_m$, $\overline{D_n^m} \subset U_n$, and $\overline{D_n^{m+1}} \subset$

D_n^m. Let $z = \{X\}$ and, for each positive integer n, $O_n = [\bigcup_{i=n}^{\infty} D_i^n] \cup \{z\}$. The sequence $\{H_n\}_{n=1}^{\infty}$ such that, for each positive integer n, $H_n = G_n \cup \{O_n\}$ is a development for $X \cup \{z\}$. Suppose V is an open subset of $X \cup \{z\}$, z is in V, and x is a point of V different from z. Since x is not a cluster point of $\{x_n\}_{n=1}^{\infty}$, there is a positive integer m such that $O_m \subset V$ and if $j \geq m$, z_j is not in $st(x,G_m)$. If $j \geq m$, $z_j \in D_j^m \in G_m$ and x is not in D_j^m; therefore, x is not in O_m and $z \in \overline{O}_{m+1} = \overline{[\bigcup_{n=m+1}^{\infty} D_n^{m+1}]} \cup \{z\} = [\bigcup_{n=m+1}^{\infty} \overline{D_n^{m+1}}] \cup \{z\}$ $[\bigcup_{n=1}^{\infty} D_n^m] \cup \{z\} .= O_m \subset V - \{x\}$. The space $X \cup \{z\}$ is regular; hence, it is a Moore space. The identity map embeds X in $X \cup \{z\}$ and $\{z_n\}_{n=1}^{\infty}$ has limit z.

<u>Theorem 2</u>. A Moore-closed space with a v-normal development is compact.

Proof. Suppose $\{G_n\}_{n=1}^{\infty}$ is a v-normal development for the Moore-closed space X. The space X is separable since it is Moore-closed. Suppose X is not perfectly separable. There is an uncountable subset K of X with no limit point [11, p. 9]. If p is in K, there is a sequence $\{R_n(p)\}_{n=1}^{\infty}$ such that $p \in R_n(p) \in G_n$ and $R_{n+1}(p) \subset R_n(p)$. There are sequences $\{K_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that $K_0 = K$ and if n is a positive integer, $x_n \in \bigcap_{p \in K_n} R_n(p)$. If n is a positive integer, let $U_n = \bigcup_{p \in K_n} R_n(p)$. Suppose $\{y_n\}_{n=1}^{\infty}$ is a sequence such that for each positive integer n, $y_n \in U_n$; $\{y_n\}_{n=1}^{\infty}$ has a cluster point y; and $\{x_n\}_{n=1}^{\infty}$ does not converge to y. There is a positive integer k and an integer $j > k$ such that x_j is not in $st(y,G_k)$. This contradicts the fact that $y \in \overline{U}_j \subset \overline{st(x_j,G_j)} \subset st(x_j,G_{j-1}) \subset st(x_j,G_k)$.

We have shown that if $\{x_n\}_{n=1}^{\infty}$ does not converge, $\{x_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ satisfy the hypothesis of the lemma and X can be embedded in a Moore space in which the image of $\{x_n\}_{n=1}^{\infty}$ converges. This is a contradiction since X is Moore-closed.

Let x denote the limit of $\{x_n\}_{n=1}^{\infty}$. There are open sets V and W such that $x \in V \subset \overline{V} \subset W \subset \overline{W} \subset X - (K \cup \{x\})$, and a sequence $\{p_n\}_{n=1}^{\infty}$ such that for each positive integer n, p_n is in $U_n - \overline{W}$. We have shown that if $\{y_n\}_{n=1}^{\infty}$ is a sequence such that for each positive integer n, $y_n \in U_n - \overline{W} \subset U_n$ and $\{y_n\}_{n=1}^{\infty}$ has a cluster point y, then $\{x_n\}_{n=1}^{\infty}$ converges to y and so, y = x. Since y is in S - W, no such sequence has a cluster point.

The sequences $\{p_n\}_{n=1}^{\infty}$ and $\{U_n - \overline{W}\}_{n=1}^{\infty}$ satisfy the hypothesis of the lemma; thus, it follows that X can be embedded in a Moore space in which the image of $\{p_n\}_{n=1}^{\infty}$ converges. Since X is Moore-closed, $\{p_n\}_{n=1}^{\infty}$ converges in X. This contradicts the fact that no sequence such as $\{p_n\}_{n=1}^{\infty}$ has a cluster point.

The space X is perfectly separable, metrizable [13, p. 8], and compact [4].

<u>Corollary</u>. If X is a Moore space with a v-normal development and M is a conditionally compact subset of X, then $\overline{M}$ is compact.

Proof. Suppose $\{G_n\}_{n=1}^{\infty}$ is a v-normal development for the Moore space X and M is a conditionally compact subset of X. If n is a positive integer, let H_n be the collection to which U belongs if, and only if, there is a member O of G_n such that $U = O \cap \overline{M}$. The sequence $\{H_n\}_{n=1}^{\infty}$ is a v-normal development for $\overline{M}$. If $\overline{M}$ is not compact, it is Moore-closed; thus, $\overline{M}$ is compact.

Definition 5. If X is a T_0-space, a function d from X^2 into the set of all real numbers is a semimetric provided that: if x is in X and y is in X, $d(x,y) = d(y,x) \geq 0$, and $d(x,y) = 0$, if, and only if, $x = y$; and the point p is a limit point of the subset C of X if, and only if, for each $\varepsilon > 0$, there is a point x in C different from p such that $d(p,x) < \varepsilon$. A T_0-space X is semimetrizable if it has a semimetric.

A regular space admits an upper semi-continuous semimetric if, and only if, it is a Moore space [3]. H. Cook [3] has shown that a space with an upper semi-continuous semimetric that is continuous in one variable has a v-normal development and that a space with a continuous semimetric has a normal development.

The Space K. The space K is a modification of Example N [2]. If we assume Martin's Axiom and the denial of the continuum hypothesis, N exists and is a subspace of K. These assumptions are not needed for the existence of K nor are they used in [2] to show that N has a semimetric continuous in one variable and that N has no v-normal development. In the construction of K and in the proof of Theorem 1, we use the methods used in [2].

The points of K are the points of the open upper half-plane in E^2 together with the points of the x-axis, X. If p is in $K - X$ and ε is a positive number, $R_\varepsilon(p) = \{p\}$; if $p = (x,0)$ is in X and ε is a positive number, $R_\varepsilon(p)$ is the bounded component of the complement in E^2 of the triangle with vertices $(x + \varepsilon,\varepsilon)$, $(x - \varepsilon,\varepsilon)$, and $(x,0)$;

together with the point p. If n is a positive integer, $G_n = \{R_\varepsilon(p) | p \varepsilon K \text{ and } \varepsilon \leq 1/n\}$. The sequence $\{G_n\}_{n=1}^{\infty}$ is a development for the Moore space K. Define the function d from K^2 into the set of all real numbers: if each of p = (x,y) and q = (u,v) is a point of K, (1) d(p,q) = 0 if p = q; (2) if 0 < y < 1 and q is in the bounded component of the complement in E^2 of the triangle with vertices (x - y,0), (x + y,0), and (x,y), or if v = 0 and x - y < u < x + y, then d(p,q) = d(q,p) = y; (3) if d(p,q) is not defined by (1) or (2), then d(p,q) = d(q,p) = 1. The function d is a semimetric for K and d is continuous in one variable.

Theorem 3. The space K has no v-normal development.

Proof. Suppose $\{H_n\}_{n=1}^{\infty}$ is a v-normal development for K. There is a positive integer and a subset A of X such that if p is in A, $st(p,H_n) \subset R_1(p)$ and the closure of A in the Euclidean topology on X contains an interval, J. There is a positive integer m and a subset B of J such that if p is in B, $R_{1/m}(p)$ is contained in some element of H_{n+1} and the closure of B in the Euclidean topology on X contains an interval. Then there is a point p = (x,0) of A and a sequence $\{q_k\}_{k=1}^{\infty}$ such that for each positive integer k, $q_k = (x_k,0)$, q_k is in B, and $0 < x_k - x < 1/km$. If k is a positive integer, the point z = (x + 3/4m,3/4m) and the point w_k = (x + 1/2km,3/4km) are in $R_{1/m}(q_k)$. The sequence $\{w_k\}_{k=1}^{\infty}$ converges to p; thus, p is in $\overline{st(z,H_{n+1})}$. The point z is not in $st(p,H_n) \subset R_1(p)$. Therefore, K has no v-normal development.

Because the space K is a Moore space, K has an upper semi-continuous semimetric. The semimetric d for K is continuous in one variable.

However, since K has no v-normal development, K admits no semimetric that is both upper semi-continuous and continuous in one variable.

SECTION 2.

The spaces D_j, C_j, and V_j. Suppose j is a positive integer greater than 1. In E^3, let P_1, P_2, ..., P_j be j open half-planes such that if $1 \leq i \leq j$, $\overline{P_i} - P_i$ is the x-axis, X. The x-axis is the union of point sets, A_1, A_2, ..., A_j, such that if $1 \leq i \leq j$, every uncountable closed subset of X intersects A_i [9]. Define

$$D_j = V_j = [\bigcup_{i=1}^{j} P_i] \cup [\bigcup_{i=2}^{j} A_i] \cup [A_1 \times \{0,1\}].$$

The space D_j. If $2 \leq i \leq j$, $\varepsilon > 0$, and a is in A_i, $R_\varepsilon(a)$ is the union of: the bounded component of $P_{i-1} - J$ where J is the circle of radius ε, containing a, lying in $P_{i-1} \cup \{a\}$; the bounded component of $P_i - K$ where K is the circle of radius ε, containing a, lying in $P_i \cup \{a\}$; and $\{a\}$. If a is in A_1 and $\varepsilon > 0$, $R_\varepsilon(a,0)$ is the union of $\{(a,0)\}$ and the bounded component of the complement in $P_1 \cup \{a\}$ of the circle of radius ε, containing a, lying in $P_1 \cup \{a\}$; $R_\varepsilon(a,1)$ is the union of $\{(a,1)\}$ and the bounded component of the complement in $P_j \cup \{a\}$ of the circle of radius ε, containing a, lying in $P_j \cup \{a\}$. If x is in $\bigcup_{i=1}^{j} P_i$, there is an open subset of E^3 with diameter less than ε, containing x, that does not intersect X; let $R_\varepsilon(x)$ denote the intersection of such a set with $\bigcup_{i=1}^{j} P_i$. The collection $\{R_\varepsilon(x) | x \in D_j \text{ and } \varepsilon > 0\}$ is a basis for the topology on D_j. The sequence $\{G_n\}_{n=1}^{\infty}$ such that if n is a positive integer, $G_n = \{R_\varepsilon(x) | x \in D_j \text{ and } 0 < \varepsilon < 1/n\}$, is a development for the Moore space D_j, which is separable and locally connected. The space D_j has the j-link property.

<u>The space V_j</u>. If $2 \leq i \leq j$, $\varepsilon > 0$, and a is in A_i, $N_\varepsilon(a)$ is the set consisting of a and all the points of $P_{i-1} \cup P_i$, within ε of a, that lie on a line containing a and forming a 45-degree angle with the x-axis. If a is in A_1 and $\varepsilon > 0$, $N_\varepsilon(a,0)$ is the set consisting of (a,0) and all points of P_1, within ε of a, that lie on a line containing a and forming a 45-degree angle with the x-axis; $N_\varepsilon(a,1)$ is the set containing of (a,1) and all points of P_j, within ε of a, that lie on a line containing a and forming a 45-degree angle with the x-axis. If x is in $\bigcup_{i=1}^{j} P_i$ and $\varepsilon > 0$, $N_\varepsilon(x) = \{x\}$. The collection $\{N_\varepsilon(x) | x \in V_j \text{ and } \varepsilon > 0\}$ is a basis for the topology on V_j. The sequence $\{G_n\}_{n=1}^{\infty}$ such that if n is a positive integer, $G_n = \{N_\varepsilon(x) | x \in V_j \text{ and } 0 < \varepsilon < 1/n\}$, is a development for the Moore space V_j. The space V_j is metacompact, has the j-link property, and has a v-normal development. If $j > 2$, V_j is continuously semimetrizable.

<u>The space C_j</u>. The space C_j is a subspace of D_j. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence such that: K_1 is a set whose only element is an interval of X of length 1; and if n is a positive integer, K_{n+1} is a collection of 2^n intervals such that each interval in K_{n+1} has length $1/3^n$, is a subset of some interval in K_n, and contains an endpoint of an interval in K_n. Let $K = \bigcup_{n=1}^{\infty} K_n$ and $\mathbf{C} = \bigcap_{n=1}^{\infty} (\bigcup K_n)$. If k is an interval in K and i is an integer, $1 \leq i \leq j$, p_{ik} is the point of P_i on the line perpendicular to X at the midpoint of k, at a distance from X equal to the length of k. The space $C_j = \{p_{ik} | k \in K \text{ and } 1 \leq i \leq j\} \cup \mathbf{C}$; C_j is separable, locally compact, and has the j-link property.

<u>Theorem 4</u>. If j is a positive integer greater than 1, neither D_j, V_j, nor C_j has the j+1-link property.

Proof. Suppose **H** is a countable family of open covers of C_j that has the j+1-link property. There is an element H of **H**, a positive number ε_1, and an uncountable subset X_1 of $A_1 \cap$ **C** such that if a is in X_1, then $R_{\varepsilon_1}(a,0) \cap C_j$ is a subset of some element of H, $R_{\varepsilon_1}(a,1) \cap C_j$ is a subset of some element of H, and (a,1) is not in $st_{j+1}((a,0),H)$. There are sequences $\{X_n\}_{n=2}^{j}$ and $\{\varepsilon_n\}_{n=2}^{j}$ such that if $2 \leq n \leq j$: $0 < \varepsilon_n < \varepsilon_{n+1}$; X_n is an uncountable subset of A_n and the closure in E^3 of X_{n-1}; and if a is in X_n, then $R_{\varepsilon_n}(a)$ $\cap$ C_j is a subset of some element of H. There is a sequence $\{x_n\}_{n=1}^{j}$ of points in the closure in E^3 of X_j such that if i is a positive integer, $1 \leq i \leq j$, x_n is in X_n and the distance in E^3 from x_n to x_1 is less than $\varepsilon_j/6$. Then, $C_j \cap R_{\varepsilon_j}(x_1,0)$ intersects $R_{\varepsilon_j}(x_2)$; $C_j \cap R_{\varepsilon_j}(x_1,1)$ intersects $R_{\varepsilon_j}(x_j)$; and if $3 \leq n \leq j$, $C_j \cap R_{\varepsilon_j}(x_{n-1})$ intersects $R_{\varepsilon_j}(x_n)$. It follows that $(x_1,1)$ is in $st_{j+1}((x_1,0),H)$, which contradicts the fact x_1 is in X_1. The space C_j does not have the j+1-link property; hence, D_j does not. The proof that V_j does not have the j+1-link property is similar.

Since V_2 does not have the 3-link property, it has no normal development. Thus, V_2 is an example of a Moore space with a v-normal development that has no normal development.

SECTION 3.

Definition 6. The statement that the space X is submetrizable means that there is a continuous one-to-one mapping of X onto a metric space [10].

Definition 7. The statement that the space X is regularly submetrizable means that there is a continuous one-to-one mapping f of X onto a metric space such that if O is an open subset of X and p is a point of O, there is an open subset U of X containing p such that the closure of f(U) is a subset of f(O) [1].

Definition 8. The statement that the space X is normally submetrizable means that there is a continuous one-to-one mapping f of X onto a metric space such that if H and K are mutually exclusive subsets of X and H and f(K) are closed, then there exist mutually exclusive subsets O and U of X containing H and K respectively such that f(O) and U are open.

In Definition 8, since f is one-to-one, f(O) and f(U) do not intersect and, because f(O) is open, $\overline{f(U)}$ does not intersect f(O). Thus, $U \subset f^{-1}(\overline{f(U)}) \subset X - O$. Since f is continuous, $\overline{U} \subset f^{-1}(\overline{f(U)})$; therefore,

$$H \subset U \subset \overline{U} \subset f^{-1}(\overline{f(U)}) \subset X - O \subset X - K.$$

Submetrizable spaces are Hausdorff; regularly submetrizable spaces are regular. Normally submetrizable spaces, however, need not be normal, as Example T in Section 4 demonstrates. We show, in the corollary to Theorem 7, that normally submetrizable spaces are completely regular. F. G. Slaughter, Jr. has observed that submetrizable spaces have the j-link property for each positive integer j.

In the theorems that follow, we say the triplet (X,f,M) is a submetric, regular submetric, or normal submetric provided X is a topological space, M is a metric space, and f is a continuous one-to-one mapping of X onto M satisfying definition 6, 7, or 8 respectively. Thus if the space X is submetrizable, there exist f and M such that (X,f,M) is a submetric.

Theorem 6. If (X,f,M) is a normal submetric, then (X,f,M) is a regular submetric.

Proof. Suppose (X,f,M) is a normal submetric, O is an open subset of X, and p is a point of O. Since X - O and {p} are mutually exclusive subsets of X and X - O and f({p}) are closed, there are mutually exclusive subsets D and E of X containing containing X - O and {p} respectively such that f(D) and E are open. The point p is in the open set E and, as the closure of f(E) does not intersect f(X - O) $\subset$ f(D), the closure of f(E) is a subset of f(O). Thus, (X,f,M) is a regular submetric.

Theorem 7. If (X,f,M) is a normal submetric, H and K are mutually exclusive subsets of X, and f(H) and K are closed, then there is a continuous mapping g of X into [0,1] such that g(H) = 0 and g(K) = 1.

Proof. Suppose (X,f,M) is a normal submetric, H and K are mutually exclusive subsets of X, $f(H)$ and K are closed, and D is the set of all nonnegative dyadic rational numbers. There is an open subset U_0 of X such that $H \subset U_0 \subset f^{-1}(\overline{f(U_0)}) \subset X - K$. If t is in D and $t > 1$, $U_t = X$. Define $U_1 = X - K$. If n is a positive integer, there is a a collection $\{U_{2i+1/2^n} | i \text{ is an integer and } 0 \leq i \leq 2^{n-1} - 1\}$ of open subsets of X such that if i is an integer and $0 \leq i \leq 2^{n-1}$,

$$f^{-1}(\overline{f(U_{2i/2^n})}) \subset U_{2i+1/2^n} \subset f^{-1}(\overline{f(U_{2i+1/2^n})}) \subset U_{2i+2/2^n}.$$

Thus, if t is in D, there is an open subset U_t of X and if s and t are in D, $s < t$, then $\overline{U}_s \subset U_t$. The function g from X into $[0,1]$ defined by $g(x) = \inf\{t | x \in U_t\}$ is continuous [8]. Since $H \subset U_0$ and $K \subset X - U_1$, $g(H) = 0$ and $g(K) = 1$.

Corollary. If (X,f,M) is a normal submetric, then X is completely regular.

SECTION 4.

The space Z. If n is an integer, P_n is an open half-plane in E^3 such that: each point of P_n has a positive third coordinate; P_n separates P_{n-1} from P_{n+1} in $\{(x,y,z)|(x,y,z) \in E^3 \text{ and } z > 0\}$; $\overline{P}_n - P_n$ is the x-axis, X; and if n and m are integers, P_n is not P_m. Let w denote a point of X. The set X - {w} is the union of a countable family $\{A_i | i \text{ is an integer}\}$ of mutually exclusive point sets such that if i is an integer, every uncountable closed subset of X intersects A_i [9, p. 514]. Define.

$$Z = [\{w\} \times \{0, 1\}] \cup \bigcup\{P_i \cup A_i | i \text{ is an integer}\}.$$

If i is an integer, p is in P_i, and n is a positive integer, $R_n(p)$ is the intersection of P_i and the bounded component of the complement in E^3 of the sphere of radius 1/n with center p. If i is an integer, x is in A_i, and n is a positive integer, $R_n(x)$ is the union of: the bounded component of P_{i-1} - S where S is the circle of radius 1/n containing x and lying in $P_{i-1} \cup \{x\}$; the bounded component of P_i - C where C is the circle of radius 1/n containing x and lying in $P_i \cup \{x\}$; and {x}. If n is a positive integer and D is the bounded component of the complement in E^3 of the sphere of radius 1/n with center w, then

$$R_n(w,0) = \{(w,0)\} \cup \bigcup[D \cup \bigcup_{i \leq -n}(P_i \cup A_i)] \text{ and}$$

$$R_n(w,1) = \{(w,1)\} \cup \bigcup[D \cup \bigcup_{n \leq i}(P_i \cup A_i)].$$

If n is a positive integer, $G_n = \{R_i(p) | p \in Z \text{ and } i \text{ is an integer} \geq n\}$.

The collection G_1 is a basis for the topology on Z and the sequence $\{G_n\}_{n=1}^{\infty}$ is a development for Z.

The separable Moore space Z has the j-link property for each positive integer j. However, by an argument similar to Jones's [7] that his space A_∞ is not completely regular at p, we can show that Z fails to be completely regular at (w,0) and at (w,1). Indeed, if f is a continuous real-valued function on Z, then f((w,0)) = f((w,1)). Thus, Z is not submetrizable. The technique used in [2] to show that the space C has no v-normal development can be used to argue that the subspace $(P_1 \cup P_2 \cup A_1 \cup A_2)$ of Z has no v-normal development; thus, Z has no v-normal development.

If a space Y is built using Younglove's method in [15], with Z as the first stage in the construction, Y will have the j-link property for each positive integer j and will be a separable, locally connected, complete Moore space on which every continuous real-valued function is constant.

The space Z - {(w,0)} is a submetrizable Moore space. It is not regularly submetrizable nor is it completely regular at (w,1); it has no v-normal development.

SECTION 5.

Theorem 8. If X is a Moore space, (X^2,f,M) is a normal submetric, and the image under f of the diagonal in X^2 is closed, then X is continuously semimetrizable.

Proof. Suppose $\{G_n\}_{n=1}^{\infty}$ is a development for the Moore space X, (X^2,f,M) is a normal submetric, Δ denotes the diagonal, $\{(x,x)|x \in X\}$, and $f(\Delta)$ is closed. If n is a positive integer, $V_n = \bigcup_{g\in G_n} (g \times g)$. There is a symmetric open subset 0_1 of X^2 such that $\Delta \subset 0_1 \subset V_1$ and 0_1 is not X^2. Suppose k is an integer greater than 1 and 0_{k-1} is a symmetric open subset of X^2 such that $\Delta \subset 0_{k-1} \subset V_{k-1}$. Since $f(\Delta)$ and $X^2 - 0_{k-1}$ are closed, there is an open subset W of X^2 such that $\Delta \subset W \subset f^{-1}(\overline{f(W)}) \subset 0_{k-1}$. Define 0_k to be a symmetric open subset of $W \cap V_k$ containing Δ.. Thus, there is a sequence $\{0_n\}_{n=1}^{\infty}$ such that if n is a positive integer, 0_n is a symmetric open subset of V_n, $\Delta \subset 0_n$, and if $n > 1$, $f^{-1}(\overline{f(0_n)}) \subset 0_{n-1}$. If (x,y) is in $X^2 - \Delta$, there is an integer k such that y is not in $st(x,G_k)$; hence, (x,y) is not in V_k and, consequently, not in 0_k. Therefore, $\bigcap_{n=1}^{\infty} 0_n = \Delta$.

It follows from Theorem 7 that if n is a positive integer, there is a continuous mapping g_n of X^2 into [0,1] such that $g_n(f^{-1}(\overline{f(0_{n+1})})) = 0$ and $g_n(X - 0_n) = 1$. If (x,y) is in X^2, define $h(x,y) = \Sigma_{n=1}^{\infty} 1/2^n \, g_n(x,y)$. The function h is continuous and maps X^2 into [0,1]. If (x,y) is in $h^{-1}(0)$, then (x,y) is in

$\bigcap_{n=1}^{\infty} O_n = \Delta$. If (x,y) is in O_k where k is an integer greater than 1, then if i is an integer, $0 < i \leq k - 1$, $g_i(x,y) = 0$. Thus, $h(x,y) \leq \Sigma_{n=k}^{\infty} 1/2^n = 1/2^{k-1}$ and $h(O_k) \subset [0,1/2^{k-1}]$.

Define the continuous mapping d of X^2 into [0,1] by: if (x,y) is in X^2, $d(x,y) = 1/2\ (h(x,y) + h(y,x))$. We will show that d is a semimetric on X^2.

The mapping d is symmetric and $d(x,y) = 0$ if , and only if, $x = y$. Suppose C is a subset of X and p is a limit point of C. Then, (p,p) is a limit point of $\{p\} \times C$ and if n is an integer greater than 1, there is a point y of C different from p such that (p,y) is in O_n. Since O_n is symmetric, (y,p) is in O_n and $h(p,y) + h(y,p) \leq 2\ (1/2^{n-1})$. Thus, $d(p,y) \leq 1/2^{n-1}$.

Suppose C is a subset of X and p is a point of X such that if n is a positive integer, there is a point x_n in C different from p and $d(x_n,p) < 1/2^n$. If k is an integer greater than 1, $g_k(X^2 - O_k) = 1$ and $h(X^2 - O_k) \geq 1/2^k$. Then, (x_k,p) is in $O_k \subset V_k$ and some region in G_k contains both x_k and p. Therefore, p is a limit point of C.

The mapping d is a continuous semimetric on X.

Example T, Theorem 9, the lemma, and Theorem 10 are due to H. Cook.

<u>The space T</u>. Let $Y = \{(x,y) | (x,y) \in E^2 \text{ and } y > 0\}$ and let X denote the x-axis. The set $T = X \cup Y$. If $\varepsilon > 0$ and p is in X, $R_\varepsilon(p)$ is the bounded component of the complement in $Y \cup \{p\}$ of the circle of radius ε, containing p and lying in $Y \cup \{p\}$, together with the point p.

If $\varepsilon > 0$ and p is in T, $S_\varepsilon(p)$ is the intersection of T and the bounded component of the complement in E^2 of the circle of radius ε, with center p, that lies in E^2. If $\varepsilon > 0$ and p is in Y, $R_\varepsilon(p) = S_\varepsilon(p)$. The collection, $\{R_\varepsilon(p) | \varepsilon > 0 \text{ and } p \in T\}$, is a basis for the topology on T.

Theorem 9 [H. Cook]. The space T is normally submetrizable.

Proof. Let the set $\{(x,y) | (x,y) \in E^2 \text{ and } y \geq 0\}$, with the Euclidean metric be the space W and let g be the map from T onto W such that if $p \in T$, $g(p) = p$. We show that the triplet (T,g,W) is a normal submetric.

Suppose H and K are mutually exclusive subsets of T and H and g(K) are closed. If p is in $K \cap X$, there is a positive integer i such that $R_{1/i}(p)$ does not intersect H. Let n_p denote the least such positive integer. If n is a positive integer, then

$$K_n = \{p | p \in K \cap X \text{ and } n_p \leq n\} \text{ and } U_n = \bigcup_{p \in K_n} R_{1/2n}(p);$$

the open subset U_n of T contains K_n and does not intersect H.

Suppose n is a positive integer, x is in W, and x is a limit point of $g(U_n)$. There is a sequence $\{p_k\}_{k=1}^{\infty}$ of points of K_n such that if k is a positive integer, the distance from x to $g(R_{1/2n}(p_k))$ in W is less than 1/k. There is a point q of $K \cap X$ such that g(q) is a limit point of $\{g(p_k) | k \text{ is a positive integer}\}$. Then, $R_{1/n}(q) \subset \bigcup_{k=1}^{\infty} R_{1/n}(p_k)$ and $\overline{g(R_{1/2n}(q))}$ contains x. Thus $n_q \leq n$ and q is in K_n. The point x is in $\overline{g(R_{1/2n}(q))} \subset g(R_{1/n}(q)) \subset W - g(H)$. The closure of $g(U_n)$ does not intersect g(H).

The open set $0 = \bigcup_{k=1}^{\infty} U_k$ contains $K \cap X$ and does not intersect H. Suppose y is a point of W in $\overline{g(0)} - g(0)$. Then, y is not in g(X), since $g(0) \cap g(X)$ is a subset of the closed set $g(K) \cap g(0)$. The point y is in g(Y) and there is a positive integer n such that the distance from y to g(X) in W is greater than 1/2n. Thus, y is not in $\overline{g(\bigcup_{k=n}^{\infty} U_k)}$. The point y is in $\overline{g(\bigcup_{k=1}^{n-1} U_k)} = \bigcup_{k=1}^{n-1} \overline{g(U_k)} \subset W - g(H)$. The open subset 0 of T contains $K \cap X$ and $\overline{g(0)}$ contains no point of g(H).

The sets g(H) and $g(K - (K \cap X))$ are mutually separated. There is a subset V of T containing $K - (K \cap X)$ such that g(V) is open and $\overline{g(V)}$ does not intersect g(H). Then, $K \subset V \cup 0$ and $H \subset T - g^{-1}(\overline{g(V \cup 0)})$, an open set. Then, $V \cup 0$ and $T - g^{-1}(\overline{g(V \cup 0)})$ are mutually exclusive subsets of T containing K and H respectively; $g(T - g^{-1}(\overline{g(V \cup 0)})) = W - \overline{g(V \cup 0)}$ and $V \cup 0$ are open. The triplet (X,g,W) is a normal submetric.

<u>Lemma</u>. If (T,f,M) is a normal submetric, 0 is a subset of T, f(0) is open, H is an uncountable subset of $0 \cap X$, and $\varepsilon > 0$, there is an uncountable subset K of H and a positive integer n such that: K is contained in a subinterval of X of length less than ε; if x is in K, $R_{1/n}(x) \subset 0$; and the image under f of the closure of K in the Euclidean topology on X is contained in $f(\overline{0}) \subset \overline{f(0)}$.

Proof. If m is a positive integer, $K_m = \{x | x \in H \text{ and } R_{1/m}(x) \subset 0\}$. There is a positive integer n and a subinterval J of X, of length less than ε, such that $K_n \cap J$ is uncountable. Let $K = K_n \cap J$. If p is a limit point of K in the Euclidean toplogy on X, then $R_{1/n}(p) - \{p\} \subset \bigcup_{x \in K} R_{1/n}(x)$, and so, $p \in \overline{0}$ and $f(p) \in \overline{f(0)}$.

<u>Theorem 10</u> [H. Cook]. T^2 is not normally submetrizable.

Proof. Suppose (T^2,F,M) is a normal submetric. Define the function f on T by: if x is in T, $f(x) = F(x,x)$. $(T,f,f(T))$ is a normal submetric. Since M is completely separable, there are subsets O_1 and O_2 of T such that $f(O_1)$ and $f(O_2)$ are open and each has diameter less than 1/2, $O_1 \cap X$ and $O_2 \cap X$ are uncountable sets, and $\overline{f(O_1)}$ and $\overline{f(O_2)}$ do not intersect. There exist uncountable sets K_1 and K_2, each contained in a subinterval of X of length less than 1/2, such that if i = 1 or 2, $K_i \subset O_i$ and the image under f of the closure of K_i in the Euclidean topology on X is contained in $\overline{f(O_i)}$.

If n is a positive integer, S_n is the collection to which α belongs, if, and only if, α is a sequence of length n and each term of α is 1 or 2. If $n > 1$ and α is in S_{n-1}, define α^1 and α^2 in S_n to be the sequences such that if $1 \leq i \leq n - 1$, $\alpha^1(i) = \alpha^2(i) = \alpha(i)$ and $\alpha^1(n) = 1$, $\alpha^2(n) = 2$. Let O_{α^1} and O_{α^2} be subsets of O_α, each containing uncountably many points of K_α, such that $f(O_{\alpha^1})$ and $f(O_{\alpha^2})$ are open and each has diameter less than $1/2^n$; and $\overline{f(O_{\alpha^1})}$ does not intersect $\overline{f(O_{\alpha^2})}$. There are uncountable sets K_{α^1} and K_{α^2}, each contained in a subinterval of X of length less than $1/2^n$, such that if i is 1 or 2, $K_{\alpha^i} \subset O_{\alpha^i} \cap K_\alpha$, and the image under f of the closure of K_{α^i} in the Euclidean topology on X is contained in $\overline{f(O_{\alpha^i})}$.

If n is a positive integer, C_n is the set to which x belongs if, and only if, there is an α in S_n such that x is in the closure of K_α in the Euclidean topology on X. Let $C = \bigcap_{n=1}^{\infty} C_n$. Suppose p is a limit point of f(C) in M. Then, p is in $\bigcup_{\alpha\varepsilon S_n} \overline{f(O_\alpha)}$ for each positive integer n. There are sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ such that if n is a positive integer: α_n is in S_n; p is in $\overline{f(O_{\alpha_n})}$; x_n is in f(C) and, for each positive integer $k \geq n$, $f^{-1}(x_k)$ is in the closure of K_{α_n} in the Euclidean topology on X; and p is a limit point of $\{x_n | n \text{ is a positive integer}\}$. Some subsequence of $\{f^{-1}(x_n)\}_{n=1}^{\infty}$ converges in the Euclidean topology on X to a point q in C. If n is a positive integer, p is in $\overline{f(O_{\alpha_n})}$ and, since q is in the closure of K_{α_n} in the Euclidean topology on X, f(q) is in $\overline{f(O_{\alpha_n})}$. Thus, the distance from p to f(q) is no greater than $1/2^n$; p = f(q) and p is in f(C). The set f(C) is closed.

There is a sequence $\{A_n\}_{n=1}^{\infty}$ such that: if n is a positive integer, A_n is an uncountable subset of C and every uncountable closed subset of f(C) intersects $f(A_n)$; if i and j are positive integers, A_i does not intersect A_j; and $C = \bigcup_{n=1}^{\infty} A_n$.

Let $\mathbf{C}$ denote $\{(x,x) | x \varepsilon C\}$ and let

$$D = \bigcup_{n=1}^{\infty} [\bigcup_{p\varepsilon A_n} (R_{1/n}(p) \times R_{1/n}(p))].$$

The subsets $\mathbf{C}$ and $T^2 - D$ of T^2 are mutually exclusive and $F(\mathbf{C})$ and $T^2 - D$ are closed. Since (T^2,F,M) is a normal submetric, there are mutually exclusive subsets E and V of T^2 containing $\mathbf{C}$ and $T^2 - D$ respectively such that E and F(V) are open. Thus, $\overline{F(E)} \subset M - F(V) \subset F(D)$. If n is a positive integer, $Q_n = \{p | p \varepsilon C \text{ and } R_{1/n}(p) \times R_{1/n}(p) \subset E\}$. There is an integer k such that Q_k is uncountable. Then, A_{2k} contains a point x such that f(x) is a limit point of $f(Q_k)$. Let β denote the ray in T perpendicular to X at the point x. Let y denote the point of β at a distance of 3/4k from x in the Euclidean metric. If n is a positive integer, w_n denotes the point of β at a distance of $1/2^n k$ from x in the Euclidean metric. In T, x is the limit of $\{w_n\}_{n=1}^{\infty}$. There is a sequence $\{z_n\}_{n=1}^{\infty}$ of points of Q_k such that if n is a positive integer, $R_{1/k}(z_n)$ contains both y and w_n. Thus, (w_n,y) is in $R_{1/k}(z_n) \times R_{1/k}(z_n) \subset E$, for each positive integer n. The point (x,y) is the limit of the sequence $\{(w_n,y)\}_{n=1}^{\infty}$ in T^2. Since F is continuous, F(x,y) is the limit of $\{F(w_n,y)\}_{n=1}^{\infty}$, each term of which is in F(E). Thus, F(x,y) is in $\overline{F(E)} \subset F(D)$ and (x,y) is in D. Since x is in A_{2k}, $\{p \mid (x,p) \in D\} = R_{1/2k}(x)$. This contradicts the fact that y is not in $R_{1/2k}(x)$. T^2 is not normally submetrizable.

<u>The space S</u>. The space S is the subspace of Z in Section 4 consisting of $P_1 \cup P_2 \cup A_1 \cup A_2$ with the relative topology. The space S is a Moore space with no v-normal development (see Section 4, page 19). Therefore, S is not continuously semimetrizable. By arguments similar to those used in Theorems 9 and 10, it can be shown that S is normally submetrizable and S^2 is not normally submetrizable.

EXAMPLE CHART

Example	Page	separable	v-normal dev	normal dev	j-link property for each j	submetrizable	regularly submetrizable	normally submetrizable	continuously submetrizable	Additional properties
K	9	0	0	0	1	1	1	1	0	has semimetric continuous in one variable
D_2	12	1	0	0	0	0	0	0	0	locally connected; does not have 3-link property
C_2	12	1	0	0	0	0	0	0	0	locally compact; does not have 3-link property
V_2	12	0	1	0	0	0	0	0	0	metacompact; does not have 3-link property
D_j	12	1	0	0	0	0	0	0	0	locally connected; does not have j+1-link property ($j > 2$)
C_j	12	1	0	0	0	0	0	0	0	locally compact; does not have j+1-link property ($j > 2$)
V_j	12	0	1	1	0	0	0	0	0	metacompact; does not have j+1-link property ($j > 2$)
Z	18	1	0	0	1	0	0	0	0	has two points that cannot be separated by a continuous real-valued function
Y	19	1	0	0	1	0	0	0	0	every continuous real-valued function is constant
Z-{(w,0)}	19	1	0	0	1	1	0	0	0	is not completely regular
T	21	1	1	1	1	1	1	1	1	T^2 is not normally submetrizable
S	26	1	0	0	1	1	1	1	0	S^2 is not normally submetrizable

BIBLIOGRAPHY

1. K. Alster and T. Przymusiński. "Normality and Martin's Axiom," to appear in Fundamenta Mathematicae.

2. H. Cook. "Cartesian products and continuous semimetrics, II," to appear in Fundamenta Mathematicae.

3. ______. "A characterization of continuously semimetrizable spaces," Set-Theoretic Topology, edited by G. M. Reed, Academic Press, New York (1977), pp. 91-95.

4. J. W. Green. "Moore-closed spaces, completeness and centered bases," General Topology and its Applications 4 (1974), pp. 297-313.

5. __________. "Moore-closed and locally Moore-closed spaces," Set-Theoretic Topology, edited by G. M. Reed, Academic Press, New York (1977), pp. 193-217.

6. R. W. Heath. "Metrizability, compactness and paracompactness in Moore spaces," Notices of the American Mathematical Society, vol. 10 (1963), p. 105.

7. F. B. Jones. "Moore spaces and uniform spaces," Proceedings of the American Mathematical Society, vol. 9 (1958), pp. 483-486.

8. J. L. Kelley. General Topology, Van Nostrand Reinhold Company, New York (1955), p. 114.

9. K. Kuratowski. Topology, Vol. I, Academic Press, New York and London (1966), p. 514.

10. H. W. Martin. "Contractibility of topological spaces onto metric spaces," Pacific Journal of Mathematics, vol. 61 (1) (1975), pp. 209-224.

11. R. L. Moore. Foundations of Point Set Theory, revised edition, American Mathematical Society Colloquium Publications, vol. 13 (1962).

12. G. M. Reed. "On chain conditions in Moore spaces," General Topology and its Applications 4 (1974), pp. 256-267.

13. G. T. Whyburn. Analytic Topology, revised edition, American Mathematical Society Colloquium Publications, vol. 28 (1971), p. 7.

14. W. A. Wilson. "On semi-metric spaces," American Journal of Mathematics, vol. 53 (1931), pp. 361-373.

15. J. N. Younglove. "A locally connected, complete Moore space on which every continuous real valued function is constant," Proceedings of the American Mathematical Society, vol. 20 (1969), pp. 527-530.